

Rigidity of eigenvalues for β ensemble in multi-cut regime

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Abstract

We prove the rigidity of eigenvalues in the bulk for β ensemble in multi-cut regime. The main method is to decompose the β ensemble in multi-cut regime to be a product of β ensembles in one-cut regime.

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1 Introduction

1.1 Background

A β ensemble is a probability measure $\mu = \mu^{(N)}$ on \mathbb{R}^N with density

$$\frac{1}{Z_\mu} \exp \left(-\frac{N\beta}{2} \sum_{i=1}^N V(\lambda_i) \right) \prod_{i < j} |\lambda_i - \lambda_j|^\beta \quad (1.1)$$

where Z_μ is the normalization constant. The function V is called the potential and Z_μ is also called the partition function. If $\beta = 1$ (resp. $\beta = 2$, $\beta = 4$), then μ is the measure induced

by the eigenvalues of a random orthogonal (resp. Hermitian, symplectic) matrix $M_{N \times N}$ with law $e^{-\frac{N\beta}{2}\text{tr}V(M_{N \times N})}dM_{N \times N}$ where $dM_{N \times N}$ is the Lebesgue measure on the set of $N \times N$ orthogonal (resp. Hermitian, symplectic) matrices. If $V(x)$ is a quadratic polynomial and $\beta > 0$, then μ can be induced from a tri-diagonal random matrix model (see [12]) and can be described by stochastic differential equations (see [22]). If $V(x) = \frac{x^2}{2}$ and $\beta = 1$ (resp. $\beta = 2, \beta = 4$), then μ is the measure induced by the eigenvalues of a random matrix in the classic GOE (resp. GUE, GSE) ensemble. For the last case, the model was well studied by Dyson, Gaudin and Mehta and the sine kernel law was obtained. See [17].

It is well known that if $V(x)$ is real analytic and increases faster than $2\ln|x|$ as $|x| \rightarrow +\infty$, then the empirical measure of μ will converge almost surely and in expectation to an equilibrium measure $\rho(t)dt$. Moreover, the support of the $\rho(t)$ is compact and is the union of a finite number of closed intervals (see, for example, Theorem 1.1 of [5]):

$$\text{supp}\rho(t) = \cup_{i=1}^q [A_i, B_i].$$

If $q = 1$, then we say that μ is in the one-cut regime. Otherwise we say that μ is in the multi-cut regime and call each $[A_i, B_i]$ a cut.

Bourgade, Erdős and Yau [6, 7, 8] studied the β ensemble in one-cut regime. They proved the rigidity of eigenvalues in the bulk (see [6, 7]) and also the rigidity of eigenvalues in the edge (see [8]). In this paper we will study the β ensemble in multi-cut regime and prove the rigidity of eigenvalues in the bulk. Bekerman [2] proved that for β ensemble in multi-cut regime, the eigenvalues near $A_2, \dots, A_q, B_1, \dots, B_{q-1}$ can jump to the adjacent cut with a positive probability. Therefore rigidity for eigenvalues near these edges does not hold.

Based on the rigidity of eigenvalues, Bourgade, Erdős and Yau proved the bulk universality (see [6, 7]) and edge universality (see [8]) for β ensemble in one-cut regime. Bekerman, Figalli and Guionnet [3] also proved the bulk and edge universality for β ensemble in one-cut regime. The method they used is an approximate transport map. Also with the method of approximate transport map, Bekerman [2] proved the bulk and edge universality for β ensemble in multi-cut regime. Shcherbina [21] proved the bulk universality for β ensemble in both one-cut and multi-cut regime by a change of variable method. Other works about universality include [10, 11, 18, 19] and they are for $\beta \in \{1, 2, 4\}$.

The fluctuation of linear statistics of eigenvalues for β ensemble is an interesting topic. It was studied in one-cut case by Johansson [16]. [8] also contains a result for the fluctuation in one-cut case (see Lemma 6.5 of [8]). In the multi-cut case, the fluctuation was studied by Borot, Guionnet [5] and Shcherbina [20]. In this paper we also obtain results about the fluctuation. See Lemma 2.6 and Theorem 9.1. The method we use is the same as Lemma 6.5 of [8].

Other interesting results about β ensemble include the asymptotic expansion of the correlators and partition function (see [4, 5] and the reference therein). This is related to Section 7.2 of this paper.

The main idea of this paper is to decompose the β ensemble in multi-cut regime to be a product of β ensembles in one-cut regime (see Section 3). We believe that using this decomposition method and the methods developed by Bourgade, Erdős and Yau [6, 7, 8] one can solve the following further problems:

- rigidity of eigenvalues near the edges A_1 and B_q ;
- bulk universality and universality near edges A_1 and B_q

with the model generalized in the following ways (as in [5] and [8]).

- V is C^4 instead of real analytic.
- V depends on N and converges to its limit $V^{(0)}$ uniformly.
- The domain of V is a neighborhood of $\cup_{i=1}^q [A_i, B_i]$ instead of \mathbb{R} .

1.2 Main result

The following lemma is well known. See, for example, Theorem 1.1 of [5].

Lemma 1.1. *If V is real analytic and $\liminf_{|x| \rightarrow \infty} \frac{V(x)}{\ln(1 + |x|)} > 2$, then:*

1. $Z_\mu < +\infty$
2. *The empirical measure $\frac{1}{N} \sum_{i=1}^N \delta(x - \lambda_i)$ converges almost surely and in expectation to an equilibrium measure. Moreover, the equilibrium measure has a continuous and compactly supported density.*
3. *The equilibrium measure $\rho(x)dx$ is the unique minimizer (in the set of probability measures on \mathbb{R}) of the functional:*

$$\nu \mapsto \int V(x) d\nu(x) - \iint \ln|x - y| d\nu(x) d\nu(y)$$

4. $\rho(x) = r(x) \prod_{j=1}^q \left(\sqrt{(x - A_j)(B_j - x)} \mathbb{1}_{[A_j, B_j]}(x) \right)$ and $r(x)$ is analytic in a neighborhood of $\cup_{j=1}^q [A_j, B_j]$. Here $[A_1, B_1], \dots, [A_q, B_q]$ are disjoint intervals.

From now on we assume that $V(x)$ satisfies the following Hypothesis 1.1.

Hypothesis 1.1. 1. V is real analytic.

2. $\liminf_{|x| \rightarrow \infty} \frac{V(x)}{\ln(1 + |x|)} > 2$.
3. $\inf_{x \in \mathbb{R}} V''(x) > -2U$ for some $U > 0$.
4. *The equilibrium measure $\rho(x)dx$ can be written as*

$$\rho(x) = r(x) \prod_{j=1}^q \left(\sqrt{(x - A_j)(B_j - x)} \mathbb{1}_{[A_j, B_j]}(x) \right)$$

and $r(x)$ is positive on $[A_i, B_i]$ for $1 \leq i \leq q$.

5. *The function $x \mapsto V(x) - 2 \int_{\mathbb{R}} \ln|x - y| \rho(y) dy$ achieves its minimum only on the support of μ :*

$$\sigma := [A_1, B_1] \cup \dots \cup [A_q, B_q].$$

Remark. According to Lemma 1.1, the model is well defined and Condition 4 makes sense.

For $1 \leq k \leq N$, define the classical location of the k th particle $\eta_k = \eta_k^{(N)}$ by

$$\eta_k = \inf\{x \in \mathbb{R} \mid \int_{-\infty}^x \rho(x) dx = \frac{k}{N}\}$$

For $1 \leq i \leq q$, define R_i to be the area of the region under the curve of $\rho(x)$ over $[A_i, B_i]$:

$$R_i = \int_{A_i}^{B_i} \rho(x) dx.$$

Obviously $\sum_{i=1}^q R_i = 1$. We make the convention that $R_0 = 0$.

Suppose

$$\Sigma^{(N)} = \{(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N \mid \lambda_1 \leq \dots \leq \lambda_N\}.$$

Suppose μ_s is a probability measure on $\Sigma^{(N)}$ with density

$$\frac{1}{Z_{\mu_s}} \exp\left(-\frac{N\beta}{2} \sum_{i=1}^N V(\lambda_i)\right) \prod_{i < j} |\lambda_i - \lambda_j|^\beta \quad (1.2)$$

where Z_{μ_s} is the normalization constant.

Our main results is Theorem 1.2.

Theorem 1.2 (Rigidity of eigenvalues in the bulk). *For any $\alpha > 0$, $\epsilon > 0$, there exist $N_0 > 0$ and $c > 0$ both depending on V , α and ϵ such that if $N > N_0$ and $1 \leq i_0 \leq q$, then*

$$\mathbb{P}^{\mu_s}(\exists k \in [(R_1 + \dots + R_{i_0-1} + \alpha)N, (R_1 + \dots + R_{i_0} - \alpha)N] \text{ such that } |\lambda_k - \eta_k| > N^{-1+\epsilon}) < \exp(-N^c).$$

Remark. Theorem 1.2 has been proved in the one-cut case, (i.e., the case that $q = 1$) by Bourgade, Erdős and Yau. (See Theorem 1.1 of [7].)

1.3 Main structure of this paper

We prove Theorem 1.2 by the following steps.

1. In section 3, we decompose μ_s to be a product of β ensembles in one-cut regime. Denote each of the latter by $\tilde{\mu}_s$.
2. In Section 4 we prove Theorem 1.2 based on the rigidity of $\tilde{\mu}_s$.
3. From Section 5 to the end of this paper we prove the rigidity of $\tilde{\mu}_s$.

One cannot directly use the methods developed by Bourgade, Erdős and Yau [7, 6, 8] to prove the rigidity of $\tilde{\mu}_s$ because $\tilde{\mu}_s$ contains a parameter c_N . It is critical to show that the speed of each convergence is independent of c_N . In the third step we use various tools including large deviation principle of empirical measure, loop equation, estimation of the fluctuation of linear statistics of eigenvalues, convexification of Hamiltonian, log Sobolev inequality and the idea of induction introduced by Bourgade, Erdős and Yau [7, 6, 8].

2 Basic results

Lemma 2.1. *Suppose the support of ρ is σ . Then σ and ρ are uniquely determined by the condition:*

1. $V(x) - 2 \int \ln |x - y| \rho(y) dy = \min_{x \in \mathbb{R}} \left(V(x) - 2 \int \ln |x - y| \rho(y) dy \right)$ for $x \in \sigma$;
2. $V(x) - 2 \int \ln |x - y| \rho(y) dy \geq \min_{x \in \mathbb{R}} \left(V(x) - 2 \int \ln |x - y| \rho(y) dy \right)$ for $x \notin \sigma$;
3. $\sigma = \text{supp}\{\rho\}$.

Lemma 2.1 is well known. See, for example, Section 1 of [19].

Lemma 2.2. *Suppose $f(x_1, \dots, x_N)$ is a symmetric function, then $\mathbb{E}^\mu(f(\lambda)) = \mathbb{E}^{\mu_s}(f(\lambda))$.*

Proof. We can rewrite the density of μ and μ_s as

$$\mu(d\lambda) = \frac{1}{Z_\mu} \exp \left(-N\beta \mathcal{H}(\lambda) \right) d\lambda, \quad \mu_s(d\lambda) = \frac{1}{Z_{\mu_s}} \exp \left(-N\beta \mathcal{H}(\lambda) \right) d\lambda \quad (2.3)$$

where $\mathcal{H}(\lambda) = \frac{1}{2} \sum V(\lambda_i) - \frac{1}{N} \sum_{i < j} \ln |\lambda_i - \lambda_j| = \frac{1}{2} \sum V(\lambda_i) - \frac{1}{2N} \sum_{i \neq j} \ln |\lambda_i - \lambda_j|$.

Noticing that \mathcal{H} is symmetric in λ and $Z_\mu = N! Z_{\mu_s}$, we complete the proof. \square

Lemma 2.3. *For any $\alpha > 0$, $\epsilon > 0$ and $1 \leq i \leq q$, there exists $c > 0$, $N_0 > 0$ both depending on V , α and ϵ such that if $N > N_0$, then*

$$\mathbb{P}^{\mu_s}(\exists k \in [(R_1 + \dots + R_{i-1} + \alpha)N, (R_1 + \dots + R_i - \alpha)N] \text{ such that } |\lambda_k - \eta_k| > \epsilon) < \exp(-N^c).$$

For the convenient of readers, we provide a proof of Lemma 2.3 in Appendix A.

Lemma 2.4. *Suppose $A \subset \mathbb{R}$ is an open set containing σ , i.e., the support of ρ . There exists $N_0 > 0$, $c > 0$ both depending on V and A such that if $N > N_0$, then*

$$\mathbb{P}^\mu(\exists k \in [1, N] \text{ such that } \lambda_k \notin A) < \exp(-cN).$$

Lemma 2.4 can be found in Remark 1.9 of [15].

Corollary 2.5. *Suppose $A \subset \mathbb{R}$ is an open set containing σ , i.e., the support of ρ . There exists $N_0 > 0$, $c > 0$ both depending on V and A such that if $N > N_0$, then*

$$\mathbb{P}^{\mu_s}(\exists k \in [1, N] \text{ such that } \lambda_k \notin A) < \exp(-cN).$$

Proof. Notice that $(\lambda_1, \dots, \lambda_N) \mapsto 1 - \prod_{i=1}^N \mathbb{1}_A(\lambda_i)$ is a symmetric function. So by Lemma 2.2 and Lemma 2.4 we have

$$\begin{aligned} \mathbb{P}^{\mu_s}(\exists k \in [1, N] \text{ such that } \lambda_k \notin A) &= \mathbb{E}^{\mu_s}(1 - \prod_{i=1}^N \mathbb{1}_A(\lambda_i)) = \mathbb{E}^\mu(1 - \prod_{i=1}^N \mathbb{1}_A(\lambda_i)) \\ &= \mathbb{P}^\mu(\exists k \in [1, N] \text{ such that } \lambda_k \notin A) \\ &\leq \exp(-cN). \end{aligned}$$

\square

Lemma 2.6. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded C^2 function with $\|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty < C_b$. Suppose $\tau > 0$. There exist $c > 0$, $N_0 > 0$ both depending on V , C_b and τ such that if $N > N_0$, then

$$\mathbb{P}^\mu \left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \right) < \exp(-N^c).$$

Lemma 2.6 can be proved in the same way as Theorem 9.1. See Section 9.

Corollary 2.7. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded C^2 function with $\|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty < C_b$. Suppose $\tau > 0$. There exist $c > 0$, $N_0 > 0$ both depending on V , C_b and τ such that if $N > N_0$, then

$$\mathbb{P}^{\mu_s} \left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \right) < \exp(-N^c).$$

Proof. Notice that $(\lambda_1, \dots, \lambda_N) \mapsto \tilde{f}(\lambda) := \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx$ is a symmetric function. So by Lemma 2.2 and Lemma 2.6 we have

$$\begin{aligned} \mathbb{P}^{\mu_s} \left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \right) &= \mathbb{E}^{\mu_s} (\mathbf{1}_{|\tilde{f}(\lambda)| > N^\tau}) = \mathbb{E}^\mu (\mathbf{1}_{|\tilde{f}(\lambda)| > N^\tau}) \\ &= \mathbb{P}^\mu \left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \right) \\ &\leq \exp(-N^c). \end{aligned}$$

□

Lemma 2.8. Suppose $i \in \{1, \dots, q\}$. Suppose S is an open interval containing $[A_i, B_i]$. Suppose $\overline{S} \cap [A_j, B_j] = \emptyset$ for all $j \neq i$. For any $\tau > 0$. There exist $c > 0$, $N_0 > 0$ both depending on V and S such that if $N > N_0$, then

$$\mathbb{P}^\mu (|\#\{j | \lambda_j \in S\} - NR_i| > N^\tau) < \exp(-N^c).$$

Proof. Suppose S' is an open interval containing S and $\overline{S'} \cap [A_j, B_j] = \emptyset$ for any $j \neq i$. Suppose f is a C^2 function satisfying:

1. $f(x) = 1$ if $x \in S$,
2. $f(x) = 0$ if $x \notin S'$,
3. $0 \leq f(x) \leq 1$ if $x \in S' \setminus S$,
4. $\|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty < \infty$

So $NR_i = N \int f(x)\rho(x)dx$ and $|\sum f(\lambda_i) - \#\{j|\lambda_j \in S\}| \leq \#\{j|\lambda_j \in S' \setminus S\}$. Thus

$$\begin{aligned} & \mathbb{P}^\mu(|\#\{j|\lambda_j \in S\} - NR_i| > N^\tau) \\ & \leq \mathbb{P}^\mu(|\#\{j|\lambda_j \in S\} - \sum f(\lambda_i)| > \frac{1}{2}N^\tau) + \mathbb{P}^\mu(|\sum f(\lambda_i) - NR_i| > \frac{1}{2}N^\tau) \\ & \leq \mathbb{P}^\mu(\#\{j|\lambda_j \in S' \setminus S\} > \frac{1}{2}N^\tau) + \mathbb{P}^\mu(|\sum f(\lambda_i) - N \int f(x)\rho(x)dx| > \frac{1}{2}N^\tau) \\ & \leq \mathbb{P}^\mu(\exists \lambda_j \in S' \setminus S) + \mathbb{P}^\mu(|\sum f(\lambda_i) - N \int f(x)\rho(x)dx| > N^{\tau/2}). \end{aligned}$$

Applying Lemma 2.4 and Lemma 2.6 we complete the proof. \square

3 Decomposition of beta ensemble

We rewrite the density of μ as

$$\frac{1}{Z_\mu} \exp\left(-N\beta\mathcal{H}(\lambda)\right) \quad (3.4)$$

where $\mathcal{H}(\lambda) = \frac{1}{2} \sum V(\lambda_i) - \frac{1}{N} \sum_{i < j} \ln |\lambda_i - \lambda_j| = \frac{1}{2} \sum V(\lambda_i) - \frac{1}{2N} \sum_{i \neq j} \ln |\lambda_i - \lambda_j|$.

Suppose $\kappa > 0$ is a small constant satisfying the following conditions.

Condition on κ

1. $0 < \kappa < 0.1$
2. $\kappa < \frac{1}{100} \min(A_2 - B_1, A_3 - B_2, \dots, A_q - B_{q-1})$
3. $r(z)$ is analytic on a neighborhood of $\{z \in \mathbb{C} | \text{dist}(z, \sigma) \leq 10\kappa\}$,
4. $r(z)$ is positive on σ and nonzero on a neighborhood of $\{z \in \mathbb{C} | \text{dist}(z, \sigma) \leq 10\kappa\}$.

Remark. The κ satisfying the above conditions exists because of the fourth condition in Hypothesis 1.1 and the fourth conclusion in Lemma 1.1.

For $1 \leq i \leq q$, set $\sigma_i = [A_i, B_i]$ and $\sigma_i(\kappa) = [A_i - \frac{\kappa}{2}, B_i + \frac{\kappa}{2}]$. Thus $\cup_{i=1}^q \sigma_i = \sigma$ and $\sigma_i(\kappa) \cap \sigma_j(\kappa) = \emptyset$ if $i \neq j$. Set $\sigma(\kappa) = \cup_{i=1}^q \sigma_i(\kappa)$.

Set $\Sigma_\kappa^{(N)} = \{\lambda \in \sigma(\kappa)^N | \lambda_1 \leq \dots \leq \lambda_N\}$.

3.1 The measure μ_κ on $\Sigma_\kappa^{(N)}$

Suppose $\mu_\kappa = \mu_\kappa^{(N)}$ is a probability measure on $\Sigma_\kappa^{(N)}$ with density

$$\frac{1}{Z_{\mu_\kappa}} \exp(-\beta N \mathcal{H}(\lambda)) \prod_{i=1}^N \mathbb{1}_{\sigma(\kappa)}(\lambda_i)$$

where Z_{μ_κ} is the normalization function. Thus μ_κ depends on V and κ .

Lemma 3.1. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded C^2 function with $\|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty < C_b$. Suppose $\tau > 0$. There exist $c > 0$, $N_0 > 0$ both depending on V , κ , C_b and τ such that if $N > N_0$, then

$$\mathbb{P}^{\mu_\kappa} \left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \right) < \exp(-N^c).$$

Proof. Suppose $\lambda = (\lambda_1, \dots, \lambda_N)$. Notice that

$$\begin{aligned} & \mathbb{P}^{\mu_s} \left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \text{ and } \lambda \in \Sigma_\kappa^{(N)} \right) \\ &= \frac{1}{Z_{\mu_s}} \int_{\Sigma_\kappa^{(N)}} e^{-\beta N \mathcal{H}} \mathbb{1}_{\left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \right)} d\lambda \\ &= \frac{Z_{\mu_\kappa}}{Z_{\mu_s}} \frac{1}{Z_{\mu_\kappa}} \int_{\Sigma_\kappa^{(N)}} e^{-\beta N \mathcal{H}} \mathbb{1}_{\left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \right)} d\lambda \\ &= \frac{Z_{\mu_\kappa}}{Z_{\mu_s}} \mathbb{P}^{\mu_\kappa} \left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \right) \\ &= \mathbb{P}^{\mu_s} \left(\Sigma_N^\kappa \right) \mathbb{P}^{\mu_\kappa} \left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \right) \end{aligned}$$

According to Lemma 2.5, there exist $C_1 = C_1(V, \kappa) > 0$ and $N_1 = N_1(V, \kappa) > 0$ such that if $N > N_1$, then $\mathbb{P}^{\mu_s} \left(\Sigma_N^\kappa \right) > 1 - \exp(-N^{C_1}) > \frac{1}{2}$ and

$$\begin{aligned} & \mathbb{P}^{\mu_\kappa} \left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \right) \\ &= \frac{1}{\mathbb{P}^{\mu_s} \left(\Sigma_N^\kappa \right)} \mathbb{P}^{\mu_s} \left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \text{ and } \lambda \in \Sigma_N^\kappa \right) \\ &\leq 2 \mathbb{P}^{\mu_s} \left(\left| \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}} f(x) \rho(x) dx \right| > N^\tau \right). \end{aligned}$$

Applying Lemma 2.7 we complete the proof. \square

Lemma 3.2. Suppose $i \in \{1, \dots, q\}$ and $\tau > 0$. There exist $c > 0$, $N_0 > 0$ both depending on V and κ and τ such that if $N > N_0$, then

$$\mathbb{P}^{\mu_\kappa} (|\#\{j | \lambda_j \in \sigma_i(\kappa)\} - NR_i| > N^\tau) < \exp(-N^c).$$

Proof. Suppose S' is an open interval containing $\sigma_i(\kappa)$ and $S' \cap \sigma_j(\kappa) = \emptyset$ for all $j \neq i$. Suppose f is a C^2 function satisfying:

1. $f(x) = 1$ if $x \in \sigma_i(\kappa)$,
2. $f(x) = 0$ if $x \notin S'$,

3. $0 \leq f(x) \leq 1$ is $x \in S' \setminus \sigma_i(\kappa)$,
4. $\|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty < \infty$.

So $NR_i = N \int f(x)\rho(x)dx$ and $\sum f(\lambda_i) = \#\{j|\lambda_j \in \sigma_i(\kappa)\}$ for all $\lambda \in \Sigma_\kappa^{(N)}$. Thus

$$\mathbb{P}^{\mu_\kappa}(|\#\{j|\lambda_j \in \sigma_i(\kappa)\} - NR_i| > N^\tau) \leq \mathbb{P}^{\mu_\kappa}(|\sum f(\lambda_i) - N \int f(x)\rho(x)dx| > N^\tau).$$

Applying Lemma 3.1 we complete the proof. \square

3.2 The measure μ_r on $\Sigma_\kappa^{(N)}$

The following measure μ_r was introduced by Shcherbina [19].

Suppose $\mu_r = \mu_r^{(N)}$ is a probability measure on $\Sigma_\kappa^{(N)}$ with density

$$\frac{1}{Z_{\mu_r}} \exp\left(-N\beta\mathcal{H}_r(\lambda)\right) \quad (3.5)$$

where

$$\mathcal{H}_r(\lambda) = \frac{1}{2} \sum_{i=1}^N V^{(r)}(\lambda_i) - \frac{1}{2N} \sum_{i \neq j} \ln |\lambda_i - \lambda_j| \sum_{\alpha=1}^q \mathbb{1}_{\sigma_\alpha(\kappa)}(\lambda_i) \mathbb{1}_{\sigma_\alpha(\kappa)}(\lambda_j) + \frac{N}{2} \Sigma^*$$

and

1. Z_{μ_r} is the normalization constant: $Z_{\mu_r} = \int_{\Sigma_\kappa^{(N)}} \exp\left(-N\beta\mathcal{H}_r(\lambda)\right) d\lambda$,
2. $V^{(r)}(x) = \sum_{\alpha=1}^q \mathbb{1}_{\sigma_\alpha(\kappa)}(x) \left(V(x) - 2 \int_{\sigma \setminus \sigma_\alpha} \rho(y) \ln |x - y| dy\right)$,
3. $\Sigma^* = \sum_{i \neq j} \int_{\sigma_i} \int_{\sigma_j} \rho(x) \rho(y) \ln |x - y| dx dy$.

By the definition of \mathcal{H} and \mathcal{H}_r , for $\lambda \in \Sigma_\kappa^{(N)}$,

$$\Delta\mathcal{H}(\lambda) := \mathcal{H}_r(\lambda) - \mathcal{H}(\lambda) = \frac{1}{2N} \sum_{i \neq j} \ln |\lambda_i - \lambda_j| \sum_{\alpha \neq \alpha'} \mathbb{1}_{\sigma_\alpha(\kappa)}(\lambda_i) \mathbb{1}_{\sigma_{\alpha'}(\kappa)}(\lambda_j) - \sum_{j=1}^N V^*(\lambda_j) + \frac{N}{2} \Sigma^*$$

where $V^*(x) = \sum_{i=1}^q \mathbb{1}_{\sigma_i(\kappa)}(x) \int_{\sigma \setminus \sigma_i} \ln |x - y| \rho(y) dy$.

Proposition 3.1. *Suppose A_1, A_2, \dots is a sequence of events on $\Sigma_\kappa^{(N)}$. Suppose there exist $C_1 > 0$, $N_1 > 0$ such that $\mu_r(A_N) \leq \exp(-N^{C_1})$ when $N \geq N_1$. Then there exist $C_2 > 0$, $N_2 > 0$ both depending on V , κ , C_1 and N_1 such that $\mu_\kappa(A_N) \leq \exp(-N^{C_2})$ when $N \geq N_2$.*

Proof. Without loss of generality, suppose $C_1 < 2$. For any $t > 0$,

$$\mu_\kappa(A_N) = \mu_\kappa(A_N \cap \{\Delta\mathcal{H} > t\}) + \mu_\kappa(A_N \cap \{\Delta\mathcal{H} \leq t\}) \leq \mu_\kappa(\Delta\mathcal{H} > t) + \frac{Z_{\mu_r}}{Z_{\mu_\kappa}} \exp(\beta N t) \mu_r(A_N)$$

By Jensen's inequality, $\ln \frac{Z^{\mu_r}}{Z^{\mu_\kappa}} = -\ln \mathbb{E}^{\mu_r}(\exp(\beta N \Delta \mathcal{H})) \leq \mathbb{E}^{\mu_r}(-\beta N \Delta \mathcal{H}) \leq \beta N |E^{\mu_r}(\Delta \mathcal{H})|$. By [19] there is $C_0 > 0$ depending on V and κ such that $|E^{\mu_r}(\Delta \mathcal{H})| \leq C_0 \cdot N^{-1}$. (See the proof of Lemma 2 of [19].)

Setting $t = N^{-1+\frac{2}{3}C_1}$ we have for $N \geq N_1$:

$$\mu_\kappa(A_N) \leq \mu_\kappa(\Delta \mathcal{H} > N^{-1+\frac{2}{3}C_1}) + \exp(\beta C_0 + \beta N^{\frac{2}{3}C_1} - N^{C_1}).$$

Now we use the method of Section 3 of [19] to estimate $\Delta \mathcal{H}$. By definition,

$$\Delta \mathcal{H} = \sum_{\alpha \neq \alpha'} \Phi(\alpha, \alpha')$$

where

$$\begin{aligned} \Phi(\alpha, \alpha') &= \frac{1}{2N} \sum_{i \neq j} \ln |\lambda_i - \lambda_j| \mathbb{1}_{\sigma_\alpha(\kappa)}(\lambda_i) \mathbb{1}_{\sigma_{\alpha'}(\kappa)}(\lambda_j) \\ &\quad - \sum_{j=1}^N \mathbb{1}_{\sigma_\alpha(\kappa)}(\lambda_j) \int_{\sigma_{\alpha'}} \ln |\lambda_j - y| \rho(y) dy + \frac{N}{2} \int_{\sigma_\alpha} \int_{\sigma_{\alpha'}} \ln |x - y| \rho(x) \rho(y) dx dy \end{aligned}$$

Let $L = B_q - A_1 + 2$. Since $0 < \kappa < \min(0.1, (A_2 - B_1)/3, (A_3 - B_2)/3, \dots, (A_q - B_{q-1})/3)$, we have $|x - y| \in (\frac{1}{3} \min(A_2 - B_1, \dots, A_q - B_{q-1}), L)$ for any $x \in \sigma_\alpha(\kappa)$, $y \in \sigma_{\alpha'}(\kappa)$ and $\alpha \neq \alpha'$. So we can construct a function $g(x)$ such that

1. $g(x)$ depends only on V and is independent of κ ,
2. $g(x)$ is smooth,
3. $g(x)$ has a period $2L$,
4. $g(x - y) = \ln |x - y|$ when ever $x \in \sigma_\alpha(\kappa)$, $y \in \sigma_{\alpha'}(\kappa)$ and $\alpha \neq \alpha'$.

By Fourier transform,

$$g(x) = \sum_{k=-\infty}^{+\infty} c_k \exp\left(\frac{k\pi x}{L}i\right) \quad \text{where} \quad c_k = \frac{1}{2L} \int_{-L}^L g(x) \exp\left(-\frac{k\pi x}{L}i\right) dx.$$

We have from the periodicity of g that for any $p \geq 1$,

$$|c_k| \leq \frac{1}{2L} \left(\frac{L}{k\pi i}\right)^p \int_{-L}^L \exp\left(-\frac{k\pi x}{L}i\right) g^{(p)}(x) dx \leq \left(\frac{L}{k\pi}\right)^p \|g^{(p)}\|_\infty.$$

Therefore when $\alpha \neq \alpha'$,

$$\begin{aligned}
\Phi(\alpha, \alpha') + \Phi(\alpha', \alpha) &= \frac{1}{N} \sum_{i \neq j} \ln |\lambda_i - \lambda_j| \mathbb{1}_{\sigma_\alpha(\kappa)}(\lambda_i) \mathbb{1}_{\sigma_{\alpha'}(\kappa)}(\lambda_j) - \sum_{j=1}^N \mathbb{1}_{\sigma_\alpha(\kappa)}(\lambda_j) \int_{\sigma_{\alpha'}} \ln |\lambda_j - y| \rho(y) dy \\
&\quad - \sum_{j=1}^N \mathbb{1}_{\sigma_{\alpha'}(\kappa)}(\lambda_j) \int_{\sigma_\alpha} \ln |\lambda_j - y| \rho(y) dy + N \int_{\sigma_\alpha} \int_{\sigma_{\alpha'}} \ln |x - y| \rho(x) \rho(y) dx dy \\
&= \frac{1}{N} \sum_{k \in \mathbb{Z}} c_k \left[\sum_{i=1}^N e^{ik\pi\lambda_i/L} \mathbb{1}_{\sigma_\alpha(\kappa)}(\lambda_i) - N \int_{\sigma_\alpha} e^{ik\pi x/L} \rho(x) dx \right] \left[\sum_{j=1}^N e^{-ik\pi\lambda_j/L} \mathbb{1}_{\sigma_{\alpha'}(\kappa)}(\lambda_j) - N \int_{\sigma_{\alpha'}} e^{-ik\pi y/L} \rho(y) dx \right] \\
&= \frac{1}{N} \sum_{k \in \mathbb{Z}} c_k I_k^\alpha I I_k^{\alpha'}
\end{aligned}$$

where

$$I_k^\alpha = \sum_{i=1}^N e^{ik\pi\lambda_i/L} \mathbb{1}_{\sigma_\alpha(\kappa)}(\lambda_i) - N \int_{\sigma_\alpha} e^{ik\pi x/L} \rho(x) dx \quad \text{and} \quad I I_k^\alpha = \sum_{j=1}^N e^{-ik\pi\lambda_j/L} \mathbb{1}_{\sigma_\alpha(\kappa)}(\lambda_j) - N \int_{\sigma_\alpha} e^{-ik\pi y/L} \rho(y) dx.$$

(It's easy to see that $I I_k^\alpha = \overline{I_k^\alpha}$.)

So $|I_k^\alpha| \leq 2N$ and $|I I_k^\alpha| \leq 2N$. We see $\Delta \mathcal{H} = \frac{1}{N} \sum_{\alpha < \alpha'} \sum_{k \in \mathbb{Z}} c_k I_k^\alpha I I_k^{\alpha'}$

Set $w = \frac{C_1}{2}$, $p > \frac{6}{C_1}$ and $p \in \mathbb{N}$. Then $\frac{2}{3}C_1 > 3 + w - wp$ and we have

$$\begin{aligned}
\left| \sum_{|k| > N^w} c_k I_k^\alpha I I_k^{\alpha'} \right| &\leq \sum_{|k| > N^w} |c_k| 4N^2 \leq 8N^2 \sum_{k > N^w} \left(\frac{L}{k\pi} \right)^p \|g^{(p)}\|_\infty \\
&\leq 8N^2 \left(\frac{L}{\pi} \right)^p \|g^{(p)}\|_\infty \int_{N^w}^\infty x^{-p} dx = \frac{8}{p-1} \left(\frac{L}{\pi} \right)^p \|g^{(p)}\|_\infty N^{2+w-wp} \\
&\leq \frac{8}{p-1} \left(\frac{L}{\pi} \right)^p \|g^{(p)}\|_\infty N^{\frac{2}{3}C_1-1}.
\end{aligned}$$

Therefore if $N > N_3(V, C_1)$, then $\left| \sum_{|k| > N^w} c_k I_k^\alpha I_k^{\alpha'} \right| < \frac{N^{\frac{2}{3}C_1}}{4q^2}$ for all α, α' and we have

$$\begin{aligned}
\mu_\kappa(\Delta\mathcal{H} > N^{-1+\frac{2}{3}C_1}) &= \mu_\kappa\left(\sum_{\alpha < \alpha'} \sum_{k \in \mathbb{Z}} c_k I_k^\alpha I_k^{\alpha'} > N^{\frac{2}{3}C_1}\right) \leq \sum_{\alpha < \alpha'} \mu_\kappa\left(\left|\sum_{k \in \mathbb{Z}} c_k I_k^\alpha I_k^{\alpha'}\right| > \frac{N^{\frac{2}{3}C_1}}{q^2}\right) \\
&\leq \sum_{\alpha < \alpha'} \mu_\kappa\left(\left|\sum_{|k| \leq N^w} c_k I_k^\alpha I_k^{\alpha'}\right| > \frac{N^{\frac{2}{3}C_1}}{2q^2}\right) \\
&\leq \sum_{\alpha < \alpha'} \mu_\kappa\left(\sum_{|k| \leq N^w} |I_k^\alpha I_k^{\alpha'}| > \frac{N^{\frac{2}{3}C_1}}{2q^2} \|g\|_\infty^{-1}\right) \quad (\text{since } |c_k| \leq \|g\|_\infty) \\
&\leq \sum_{\alpha < \alpha'} \sum_{|k| \leq N^w} \mu_\kappa\left(|I_k^\alpha I_k^{\alpha'}| > \frac{N^{\frac{2}{3}C_1-w}}{6q^2} \|g\|_\infty^{-1}\right) \\
&\leq \sum_{\alpha < \alpha'} \sum_{|k| \leq N^w} \left[\mu_\kappa\left(|I_k^\alpha| > \frac{1}{\sqrt{6\|g\|_\infty}q} N^{C_1/12}\right) + \mu_\kappa\left(|I_k^{\alpha'}| > \frac{1}{\sqrt{6\|g\|_\infty}q} N^{C_1/12}\right) \right] \quad (\text{since } w = C_1/2)
\end{aligned}$$

Suppose $h_{\alpha, \kappa} : \mathbb{R} \rightarrow \mathbb{C}$ is a function satisfying

1. the real and imaginary parts of $h_{\alpha, \kappa}$ are both C^2 ,
2. $\|\operatorname{Re} h_{\alpha, \kappa}\|_\infty + \|(\operatorname{Re} h_{\alpha, \kappa})'\|_\infty + \|(\operatorname{Re} h_{\alpha, \kappa})''\|_\infty < \infty$,
 $\|\operatorname{Im} h_{\alpha, \kappa}\|_\infty + \|(\operatorname{Im} h_{\alpha, \kappa})'\|_\infty + \|(\operatorname{Im} h_{\alpha, \kappa})''\|_\infty < \infty$
3. $h_{\alpha, \kappa}(x) = e^{ik\pi x/L}$, whenever $x \in \sigma_\alpha(\kappa)$; $h_{\alpha, \kappa}(y) = 0$ whenever $x \in \sigma_{\alpha'}(\kappa)$ and $\alpha' \neq \alpha$.

Using Lemma 3.1 for the real and imaginary parts of $h_{\alpha, \kappa}$, we have $N_4 > 0$ and $c_* > 0$ both depending on V, κ and C_1 such that that if $N > N_4$, then

$$\begin{aligned}
\mu_\kappa\left(|I_k^\alpha| > \frac{1}{\sqrt{6\|g\|_\infty}q} N^{C_1/12}\right) &= \mu_\kappa\left(\left|\sum h_{\alpha, \kappa}(\lambda_i) - N \int h_{\alpha, \kappa}(x) \rho(x) dx\right| > \frac{1}{\sqrt{6\|g\|_\infty}q} N^{C_1/12}\right) \\
&\leq \exp(-N^{c_*}).
\end{aligned}$$

We can estimate $\mu_\kappa\left(|II_k^{\alpha'}| > \frac{1}{\sqrt{6\|g\|_\infty}q} N^{C_1/12}\right)$ similarly. So if $N > N_2(V, \kappa, C_1, N_1)$, then

$$\begin{aligned}
\mu_\kappa(A_N) &\leq \mu_\kappa(\Delta\mathcal{H} > N^{-1+\frac{2}{3}C_1}) + \exp(\beta C_0 + \beta N^{\frac{2}{3}C_1} - N^{C_1}) \\
&\leq q^2(2N^w + 1) \cdot 2 \exp(-N^{c_*}) + \exp(\beta C_0 + \beta N^{\frac{2}{3}C_1} - N^{C_1}) \\
&\leq \exp(-N^{C_2})
\end{aligned}$$

for some $C_2(V, \kappa, C_1, N_1) > 0$.

□

3.3 The measures $\nu^{(i, N, c_N)}$ on $\sigma_i(\kappa)$ and $\nu_s^{(i, N, c_N)}$ on $\bar{\Sigma}_\kappa^{(N)}(i)$

Set $\bar{\Sigma}_\kappa^{(N)}(i) := \{(\lambda_1, \dots, \lambda_N) | \lambda_1 \leq \dots \leq \lambda_N, \lambda_i \in \sigma_i(\kappa), \forall 1 \leq i \leq N\}$. Suppose $c_N > 0$.

- Set $\nu^{(i,N,c_N)}$ to be a probability measure on $\sigma_i(\kappa)^N$ with density

$$\frac{1}{Z_{\nu^{(i,N,c_N)}}} \exp \left(-\frac{N\beta}{2} \sum_{j=1}^N c_N \frac{1}{R_i} [V(\lambda_j) - 2 \int_{\sigma \setminus \sigma_i} \ln |\lambda_j - y| \rho(y) dy] \right) \prod_{1 \leq u < v \leq N} |\lambda_u - \lambda_v|^\beta \quad (3.6)$$

where $Z_{\nu^{(i,N,c_N)}}$ is the normalization constant.

- Set $\nu_s^{(i,N,c_N)}$ to be a probability measure on $\bar{\Sigma}_\kappa^{(N)}(i)$ with density

$$\frac{1}{Z_{\nu_s^{(i,N,c_N)}}} \exp \left(-\frac{N\beta}{2} \sum_{j=1}^N c_N \frac{1}{R_i} [V(\lambda_j) - 2 \int_{\sigma \setminus \sigma_i} \ln |\lambda_j - y| \rho(y) dy] \right) \prod_{1 \leq u < v \leq N} |\lambda_u - \lambda_v|^\beta \quad (3.7)$$

where $Z_{\nu_s^{(i,N,c_N)}}$ is the normalization constant.

Lemma 3.3. Suppose $f(x_1, \dots, x_N)$ is a symmetric function, then $\mathbb{E}^{\nu^{(i,N,c_N)}}(f(\lambda)) = \mathbb{E}^{\nu_s^{(i,N,c_N)}}(f(\lambda))$.

Proof. Noticing that $Z_{\nu^{(i,N,c_N)}} = N! Z_{\nu_s^{(i,N,c_N)}}$ and that

$$(\lambda_1, \dots, \lambda_N) \mapsto \exp \left(-\frac{N\beta}{2} \sum_{j=1}^N c_N \frac{1}{R_i} [V(\lambda_j) - 2 \int_{\sigma \setminus \sigma_i} \ln |\lambda_j - y| \rho(y) dy] \right) \prod_{1 \leq u < v \leq N} |\lambda_u - \lambda_v|^\beta$$

is a symmetric function, we complete the proof. \square

Suppose $\lim_{N \rightarrow \infty} c_N = 1$. According to Theorem 1.1 of [5], the empirical measure

$$\frac{1}{N} \sum_{j=1}^N \delta(x - \lambda_j)$$

of $\nu^{(i,N,c_N)}$ converges almost surely and in expectation to

$$\frac{1}{R_i} \rho(x) \mathbb{1}_{([A_i, B_i])}(x) dx$$

as $N \rightarrow \infty$. In other words, for every continuous and bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}^{\nu^{(i,N,c_N)}} \sum_{j=1}^N f(\lambda_j) = \int_{\mathbb{R}} \frac{1}{R_i} \rho(x) \mathbb{1}_{([A_i, B_i])}(x) f(x) dx.$$

According to Lemma 3.3 we also have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}^{\nu_s^{(i,N,c_N)}} \sum_{j=1}^N f(\lambda_j) = \int_{\mathbb{R}} \frac{1}{R_i} \rho(x) \mathbb{1}_{([A_i, B_i])}(x) f(x) dx$$

provided $\lim_{N \rightarrow \infty} c_N = 1$ and f is continuous and bounded.

Define the classical position of the k -th particle under $\nu^{(i,N,c_N)}$ by $\theta(i, N, k)$ by

$$\int_{A_i}^{\theta(i,N,k)} \frac{1}{R_i} \rho(x) dx = \frac{k}{N} \quad (1 \leq k \leq N). \quad (3.8)$$

Suppose $t_1 > P_1 > 0$. For $k \in \{1, 2, \dots\}$, define $a_k = \frac{1}{2}(\frac{3}{4})^{k-1}$, $P_k = P_1 \times 0.2^{k-1}$ and $t_{k+1} = 2t_k + 1.6P_k$.

Lemma 3.4. *Suppose $1 \leq i \leq q$. Suppose there is $0 < \epsilon_0 < 0.01$ such that $|c_N - 1| \leq N^{-1+\epsilon_0}$ for all $N \geq 1$. Suppose $\alpha > 0$ and $k \in \mathbb{N}$. If $P_k > 20$ and $0.4P_r\epsilon_0 < \frac{3}{4}a_r$ for all $r \in \{1, \dots, k-1\}$, then there exists $N_0 > 0$ depending on V , κ , ϵ_0 , α and k , but independent of $\{c_N\}$ (as long as the above two conditions are satisfied) such that if $N > N_0$, then*

$$\mathbb{P}^{\nu^{(i,N,c_N)}} \left(\exists j \in [\alpha N, (1-\alpha)N] \text{ st. } |\lambda_j - \theta(i, N, j)| > N^{-1+a_k+t_k\epsilon_0} \right) \leq \exp(-N^{P_k\epsilon_0}).$$

Proof. Recall that $\liminf_{|x| \rightarrow \infty} \frac{V(x)}{\ln(1+|x|)} > 2$. Thus

$$\lim_{|x| \rightarrow +\infty} |V(\lambda_j) - 2 \int_{\sigma \setminus \sigma_i} \ln |\lambda_j - y| \rho(y) dy| = +\infty.$$

Set

$$C_p = 6 \max_{x \in \sigma_i(\kappa)} \int_{A_i}^{B_i} \frac{1}{R_i} \rho(y) \ln |x - y| dy + 4 \ln(B_i - A_i + 2) + 3 - \min_{x \in \mathbb{R}} \frac{1}{R_i} \left| V(x) - 2 \int_{\sigma \setminus \sigma_i} \ln |x - y| \rho(y) dy \right|.$$

So for any $x \in \mathbb{R}$,

$$\frac{1}{R_i} [V(x) - 2 \int_{\sigma \setminus \sigma_i} \ln |x - y| \rho(y) dy] + C_p > 6 \max_{x \in \sigma_i(\kappa)} \int_{A_i}^{B_i} \frac{1}{R_i} \rho(y) \ln |x - y| dy + 4 \ln(B_i - A_i + 2) + 2.$$

We can rewrite $\nu^{(i,N,c_N)}$ and $\nu_s^{(i,N,c_N)}$ as

•

$$\begin{aligned} & \nu^{(i,N,c_N)}(dx) \\ &= \frac{1}{\bar{Z}_{\nu^{(i,N,c_N)}}} \exp \left(-\frac{N\beta}{2} \sum_{j=1}^N c_N \left[\frac{1}{R_i} [V(\lambda_j) - 2 \int_{\sigma \setminus \sigma_i} \ln |\lambda_j - y| \rho(y) dy] + C_p \right] \right) \prod_{1 \leq u < v \leq N} |\lambda_u - \lambda_v|^\beta dx \end{aligned}$$

where $\bar{Z}_{\nu^{(i,N,c_N)}}$ is the normalization constant.

•

$$\begin{aligned} & \nu_s^{(i,N,c_N)}(dx) \\ &= \frac{1}{\bar{Z}_{\nu_s^{(i,N,c_N)}}} \exp \left(-\frac{N\beta}{2} \sum_{j=1}^N c_N \left[\frac{1}{R_i} [V(\lambda_j) - 2 \int_{\sigma \setminus \sigma_i} \ln |\lambda_j - y| \rho(y) dy] + C_p \right] \right) \prod_{1 \leq u < v \leq N} |\lambda_u - \lambda_v|^\beta dx \end{aligned}$$

where $\bar{Z}_{\nu_s^{(i,N,c_N)}}$ is the normalization constant.

We will use Theorem 5.3 for $\nu_s^{(i,N,c_N)}$ with

- $\tilde{\mu}_s = \nu_s^{(i,N,\{c_1,c_2,\dots\})}$, $\tilde{\mu} = \nu^{(i,N,\{c_1,c_2,\dots\})}$
- $[c, d] = [A_i, B_i]$, $[a, b] = \sigma_i(\kappa) = [A_i - \frac{1}{2}\kappa, B_i + \frac{1}{2}\kappa]$.
- $\tilde{\rho}(x) = \frac{1}{R_i} \rho(x) \mathbb{1}_{([A_i, B_i])}(x)$
-

$$W_L = \begin{cases} B_{i-1} & \text{if } i > 1, \\ A_1 - 100 & \text{if } i = 1, \end{cases} \quad W_R = \begin{cases} A_{i+1} & \text{if } i < q, \\ B_q + 100 & \text{if } i = q. \end{cases}$$

- $V_p(x) = \frac{1}{R_i} [V(x) - 2 \int_{\sigma \setminus \sigma_i} \ln |x - y| \rho(y) dy] + C_p$.

Because of the conditions in Section 1.1, the conditions on κ , the conditions of this Lemma and the definition of C_p , all the conditions in the Hypothesis 5.1 in Section 5 are satisfied except that

1. there exists $U_p > 0$ such that $V_p''(x) \geq -2U_p$ on a neighborhood of $\sigma_i(\kappa)$,
2. $V_p \in C_0(\mathbb{R})$,
3. V_p is analytic on $\mathbb{C} \setminus ((-\infty, W_L] \cup [W_R, +\infty))$,
- 4.

$$\frac{1}{2} V_p(x) - \int_a^b \ln |x - t| \tilde{\rho}(t) dt \begin{cases} = \inf_{x \in [a, b]} \left(\frac{1}{2} V_p(x) - \int_a^b \ln |x - t| \tilde{\rho}(t) dt \right) & \text{if } x \in [c, d] \\ > \inf_{x \in [a, b]} \left(\frac{1}{2} V_p(x) - \int_a^b \ln |x - t| \tilde{\rho}(t) dt \right) & \text{if } x \in [a, b] \setminus [c, d] \end{cases}$$

To check the first one, we compute directly:

$$V_p''(x) = \frac{1}{R_i} [V''(x) + 2 \int_{\sigma \setminus \sigma_i} \frac{1}{(x - y)^2} \rho(y) dy] \geq \frac{1}{R_i} V''(x) \geq -2 \frac{U}{R_i}.$$

The second condition is ensured by the property of convolution.

For the third condition, we only have to notice that if $j \neq i$, then the function $x \mapsto \int_{A_j}^{B_j} \rho(y) \ln |x - y| dy$ can be analytically extended either to $\mathbb{C} \setminus (-\infty, B_j]$ or to $\mathbb{C} \setminus [A_j, +\infty)$.

The fourth condition comes from the fact that $x \mapsto V(x) - 2 \int_{\mathbb{R}} \ln |x - y| \rho(y) dy$ achieves its minimum only on $[A_1, B_1] \cup \dots \cup [A_q, B_q]$:

$$\begin{aligned} \frac{1}{2} V_p(x) - \int_a^b \ln |x - t| \tilde{\rho}(t) dt &= \frac{1}{2} \frac{1}{R_i} [V(x) - 2 \int_{\sigma \setminus \sigma_i} \ln |x - y| \rho(y) dy] + C_p - \int_{\sigma_i} \ln |x - y| \frac{1}{R_i} \rho(y) dy \\ &= \frac{1}{2R_i} [V_p(x) - 2 \int_{\sigma} \ln |x - t| \tilde{\rho}(t) dt] + C_p. \end{aligned}$$

The proof is complete by applying Theorem 5.3. □

4 Proof of the main theorem

In this section we prove the main theorem, i.e., Theorem 1.2.

Suppose $\alpha_* > 0$, $\epsilon_* \in (0, 0.01)$, and $1 \leq i_0 \leq q$. Set

$$A_N = \{\lambda \in \Sigma_\kappa^{(N)} \mid \exists k \in [(R_1 + \dots + R_{i_0-1} + \alpha_*)N, (R_1 + \dots + R_{i_0} - \alpha_*)N] \text{ such that } |\lambda_k - \eta_k| > N^{-1+\epsilon_*}\};$$

$$\Omega_N = \{\lambda \in \Sigma_\kappa^{(N)} \mid |\#\{i \mid \lambda_i \in \sigma_i(\kappa)\} - NR_i| \leq N^{\frac{1}{2}\epsilon_*}, \forall 1 \leq i \leq q\}.$$

By Lemma 3.2, there are $C_1 > 0$ and $N_1 > 0$ both depending on V , κ and ϵ_* such that if $N > N_1$, then

$$\mathbb{P}^{\mu_\kappa}(\Omega_N^c) \leq \exp(-N^{C_1}). \quad (4.9)$$

Notice that

$$\mathbb{P}^{\mu_r}(A_N \cap \Omega_N) = \frac{\int_{\Sigma_\kappa^{(N)}} \exp(-N\beta\mathcal{H}_r(\lambda)) \mathbb{1}_{A_N \cap \Omega_N}(\lambda) d\lambda}{\int_{\Sigma_\kappa^{(N)}} \exp(-N\beta\mathcal{H}_r(\lambda)) d\lambda} = \frac{\sum_{k_1+\dots+k_q=N} Q_{(k_1,\dots,k_q)} \cdot \Phi(k_1,\dots,k_q)}{\sum_{k_1+\dots+k_q=N} Q_{(k_1,\dots,k_q)}}$$

where

$$Q_{(k_1,\dots,k_q)} = \int_{\Sigma_\kappa^{(N)}} \prod_{j=1}^{k_1} \mathbb{1}_{\sigma_1(\kappa)}(\lambda_j) \cdots \prod_{j=k_1+\dots+k_{q-1}+1}^N \mathbb{1}_{\sigma_q(\kappa)}(\lambda_j) \exp\left(-N\beta\mathcal{H}_r(\lambda)\right) d\lambda$$

and

$$\Phi(k_1,\dots,k_q) = Q_{(k_1,\dots,k_q)}^{-1} \int_{\Sigma_\kappa^{(N)}} \prod_{j=1}^{k_1} \mathbb{1}_{\sigma_1(\kappa)}(\lambda_j) \cdots \prod_{j=k_1+\dots+k_{q-1}+1}^N \mathbb{1}_{\sigma_q(\kappa)}(\lambda_j) \exp\left(-N\beta\mathcal{H}_r(\lambda)\right) \mathbb{1}_{A_N \cap \Omega_N}(\lambda) d\lambda.$$

Remark. We learnt the idea of considering the q -tuple (k_1, \dots, k_q) from Section 3 of [19].

From the definition of \mathcal{H}_r , we see that in the integral of $Q_{(k_1,\dots,k_q)}$ the particles in different cuts don't have intersection. Thus the integral can be written as a product of integrals over domains with lower dimensions, i.e., $Q_{(k_1,\dots,k_q)} = Q_{(k_1,\dots,k_q)}^{(1)} \cdots Q_{(k_1,\dots,k_q)}^{(q)} \cdot \exp(-\frac{\beta}{2}N^2\Sigma^*)$ where

$$Q_{(k_1,\dots,k_q)}^{(i)} = \int_{\bar{\Sigma}_\kappa^{(k_i)}(i)} \exp\left(-\frac{N\beta}{2} \sum_{j=1}^{k_i} (V(\lambda_j) - 2 \int_{\sigma \setminus \sigma_i} \ln|\lambda_j - y| \rho(y) dy)\right) \prod_{u < v} |\lambda_u - \lambda_v|^\beta d\lambda_1 \cdots d\lambda_{k_i}$$

and $\bar{\Sigma}_\kappa^{(n)}(i) = \{(\lambda_1, \dots, \lambda_n) \mid \lambda_1 \leq \dots \leq \lambda_n, \lambda_i \in \sigma_i(\kappa), \forall 1 \leq i \leq n\}$, as defined before.

Now consider $\Phi(k_1, \dots, k_q)$. If

$$\left[\prod_{j=1}^{k_1} \mathbb{1}_{\sigma_1(\kappa)}(\lambda_j) \cdots \prod_{j=k_1+\dots+k_{q-1}+1}^N \mathbb{1}_{\sigma_q(\kappa)}(\lambda_j) \right] \mathbb{1}_{A_N \cap \Omega_N}(\lambda) = 1, \quad (4.10)$$

then

1. there exists $k \in [(R_1 + \dots + R_{i_0-1} + \alpha_*)N, (R_1 + \dots + R_{i_0} - \alpha_*)N]$ such that $|\lambda_k - \eta_k| > N^{-1+\epsilon_*}$;
2. $k_1 = \#\{i \in [1, N] | \lambda_i \in \sigma_1(\kappa)\}, \dots, k_q = \#\{i \in [1, N] | \lambda_i \in \sigma_q(\kappa)\}$;
3. $|\#\{i \in [1, N] | \lambda_i \in \sigma_j(\kappa)\} - NR_j| \leq N^{\frac{1}{2}\epsilon_*}$ for all $1 \leq j \leq q$.

So there exist $N_2 = N_2(V, \alpha_*)$ such that if $N > N_2$ and (4.10) is true, then

$$[(R_1 + \dots + R_{i_0-1} + \alpha_*)N, (R_1 + \dots + R_{i_0} - \alpha_*)N] \subset [k_1 + \dots + k_{i_0-1} + \frac{\alpha_*}{2R_{i_0}}k_{i_0}, k_1 + \dots + k_{i_0} - \frac{\alpha_*}{2R_{i_0}}k_{i_0}],$$

$$[k_1 + \dots + k_{i_0-1} + \frac{\alpha_*}{2R_{i_0}}k_{i_0}, k_1 + \dots + k_{i_0} - \frac{\alpha_*}{2R_{i_0}}k_{i_0}] \subset [(R_1 + \dots + R_{i_0-1} + \frac{\alpha_*}{3})N, (R_1 + \dots + R_{i_0} - \frac{\alpha_*}{3})N]$$

and there must be $k \in [k_1 + \dots + k_{i_0-1} + \frac{\alpha_*}{2R_{i_0}}k_{i_0}, k_1 + \dots + k_{i_0} - \frac{\alpha_*}{2R_{i_0}}k_{i_0}]$ such that $|\lambda_k - \eta_k| > N^{-1+\epsilon_*}$.

Since $\rho(x) > 0$ on $[A_{i_0}, B_{i_0}]$, there must be $M = M(V, \alpha_*) > 0$ such that if $k \in [k_1 + \dots + k_{i_0-1} + \frac{\alpha_*}{2R_{i_0}}k_{i_0}, k_1 + \dots + k_{i_0} - \frac{\alpha_*}{2R_{i_0}}k_{i_0}]$, then both η_k and $\theta(i_0, k_{i_0}, k - (k_1 + \dots + k_{i_0-1}))$ (which is defined in (3.8)) are in $[A_{i_0} + M, B_{i_0} - M]$. Set $\mathcal{M} = \min_{x \in [A_{i_0} + M, B_{i_0} - M]} \rho(x) > 0$.

Set $\bar{k} = k - (k_1 + \dots + k_{i_0-1})$.

So if $N > N_2$, $k \in [k_1 + \dots + k_{i_0-1} + \frac{\alpha_*}{2R_{i_0}}k_{i_0}, k_1 + \dots + k_{i_0} - \frac{\alpha_*}{2R_{i_0}}k_{i_0}]$ and (4.10) is true, then $\bar{k} \in [\frac{\alpha_*}{2R_{i_0}}k_{i_0}, k_{i_0} - \frac{\alpha_*}{2R_{i_0}}k_{i_0}]$ and

$$\begin{aligned} |\eta_k - \theta(i_0, k_{i_0}, \bar{k})| \frac{\mathcal{M}}{R_{i_0}} &\leq \left| \int_{\eta_k}^{\theta(i_0, k_{i_0}, \bar{k})} \frac{1}{R_{i_0}} \rho(x) dx \right| = \left| \int_{A_{i_0}}^{\theta(i_0, k_{i_0}, \bar{k})} \frac{1}{R_{i_0}} \rho(x) dx - \int_{A_{i_0}}^{\eta_k} \frac{1}{R_{i_0}} \rho(x) dx \right| \\ &= \left| \frac{\bar{k}}{k_{i_0}} - \frac{1}{R_{i_0}} \left(\frac{k}{N} - (R_1 + \dots + R_{i_0-1}) \right) \right| = \left| \frac{\bar{k}(NR_{i_0} - k_{i_0})}{NR_{i_0}k_{i_0}} + \frac{1}{NR_{i_0}} ((NR_1 - k_1) + \dots + (NR_{i_0-1} - k_{i_0-1})) \right| \\ &\leq \left| \frac{(NR_{i_0} - k_{i_0})}{NR_{i_0}} \right| + \frac{1}{NR_{i_0}} (|NR_1 - k_1| + \dots + |NR_{i_0-1} - k_{i_0-1}|) \leq \frac{i_0}{R_{i_0}} N^{-1+\frac{1}{2}\epsilon_*} \\ &\leq \frac{q}{R_{i_0}} N^{-1+\frac{1}{2}\epsilon_*}, \end{aligned}$$

thus

$$|\eta_k - \theta(i_0, k_{i_0}, \bar{k})| \leq \frac{q}{\mathcal{M}} N^{-1+\frac{1}{2}\epsilon_*}$$

Therefor there exists $N_3 = N_3(V, \alpha_*, \epsilon_*)$ such that if $N > N_3$ and (4.10) is true, then

- there must be $k \in [k_1 + \dots + k_{i_0-1} + \frac{\alpha_*}{2R_{i_0}}k_{i_0}, k_1 + \dots + k_{i_0} - \frac{\alpha_*}{2R_{i_0}}k_{i_0}]$ such that

$$|\lambda_k - \theta(i_0, k_{i_0}, \bar{k})| \geq |\lambda_k - \eta_k| - |\eta_k - \theta(i_0, k_{i_0}, \bar{k})| \geq N^{-1+\epsilon_*} - \frac{q}{\mathcal{M}} N^{-1+\frac{1}{2}\epsilon_*} > N^{-1+\frac{2}{3}\epsilon_*} > k_{i_0}^{-1+\frac{1}{2}\epsilon_*}.$$

- $|\frac{NR_i}{k_i} - 1| \leq k_{i_0}^{-1+\epsilon_*}$ for all $1 \leq i \leq q$.

So when $N > N_3$,

$$\left[\prod_{j=1}^{k_1} \mathbb{1}_{\sigma_1(\kappa)}(\lambda_j) \cdots \prod_{j=k_1+\dots+k_{q-1}+1}^N \mathbb{1}_{\sigma_q(\kappa)}(\lambda_j) \right] \mathbb{1}_{A_N \cap \Omega_N}(\lambda) \leq \left[\prod_{j=1}^{k_1} \mathbb{1}_{\sigma_1(\kappa)}(\lambda_j) \cdots \prod_{j=k_1+\dots+k_{q-1}+1}^N \mathbb{1}_{\sigma_q(\kappa)}(\lambda_j) \right] \mathbb{1}_{\Omega'}(\lambda)$$

where

$$\Omega' = \{ \lambda \in \Sigma_{\kappa}^{(N)} | \exists k \in [k_1 + \dots + k_{i_0-1} + \frac{\alpha_*}{2R_{i_0}} k_{i_0}, k_1 + \dots + k_{i_0} - \frac{\alpha_*}{2R_{i_0}} k_{i_0}] \text{ st. } |\lambda_k - \theta(i_0, k_{i_0}, \bar{k})| > k_{i_0}^{-1+\frac{1}{2}\epsilon_*} \}.$$

$$\text{Let } \Omega'' = \{ \lambda \in \bar{\Sigma}_{\kappa}^{(k_{i_0})}(i_0) | \exists k \in [\frac{\alpha_*}{2R_{i_0}} k_{i_0}, k_{i_0} - \frac{\alpha_*}{2R_{i_0}} k_{i_0}] \text{ st. } |\lambda_k - \theta(i_0, k_{i_0}, k)| > k_{i_0}^{-1+\frac{1}{2}\epsilon_*} \}.$$

For $\lambda \in \Sigma_{\kappa}^{(N)}$, let $\bar{\lambda} = (\lambda_{k_1+\dots+k_{i_0-1}+1}, \dots, \lambda_{k_1+\dots+k_{i_0}})$. It is easy to check that if $\lambda \in \Omega'$, then $\bar{\lambda} \in \Omega''$

Set $\mathbf{k}_{i_0} = \{k_1 + \dots + k_{i_0-1} + 1, \dots, k_1 + \dots + k_{i_0}\}$

When $N > N_3$ we have

$$\begin{aligned} & \int_{\Sigma_{\kappa}^{(N)}} \prod_{j=1}^{k_1} \mathbb{1}_{\sigma_1(\kappa)}(\lambda_j) \cdots \prod_{j=k_1+\dots+k_{q-1}+1}^N \mathbb{1}_{\sigma_q(\kappa)}(\lambda_j) \exp\left(-\beta N \mathcal{H}_r(\lambda)\right) \mathbb{1}_{A_N \cap \Omega_N}(\lambda) d\lambda \\ & \leq Q_{(k_1, \dots, k_q)}^{(1)} \cdots Q_{(k_1, \dots, k_q)}^{(i_0-1)} \cdot Q_{(k_1, \dots, k_q)}^{(i_0+1)} \cdots Q_{(k_1, \dots, k_q)}^{(q)} \cdot \exp\left(-\frac{\beta}{2} N^2 \Sigma^*\right) \\ & \quad \cdot \int_{\bar{\Sigma}_{\kappa}^{(k_{i_0})}(i_0)} \exp\left(-\frac{N\beta}{2} \sum_{j \in \mathbf{k}_{i_0}} (V(\lambda_j) - 2 \int_{\sigma \setminus \sigma_{i_0}} \ln |\lambda_j - y| \rho(y) dy)\right) \prod_{\substack{u < v \\ u, v \in \mathbf{k}_{i_0}}} |\lambda_u - \lambda_v|^{\beta} \mathbb{1}_{\Omega''}(\bar{\lambda}) \prod_{j \in \mathbf{k}_{i_0}} d\lambda_j \end{aligned}$$

and

$$\begin{aligned} & \Phi(k_1, \dots, k_q) \\ & \int_{\bar{\Sigma}_{\kappa}^{(k_{i_0})}(i_0)} \exp\left(-\frac{N\beta}{2} \sum_{j \in \mathbf{k}_{i_0}} (V(\lambda_j) - 2 \int_{\sigma \setminus \sigma_{i_0}} \ln |\lambda_j - y| \rho(y) dy)\right) \prod_{\substack{u < v \\ u, v \in \mathbf{k}_{i_0}}} |\lambda_u - \lambda_v|^{\beta} \mathbb{1}_{\Omega''}(\bar{\lambda}) \prod_{j \in \mathbf{k}_{i_0}} d\lambda_j \\ & \leq \frac{Q_{(k_1, \dots, k_q)}^{(i_0)}}{Q_{(k_1, \dots, k_q)}^{(i_0+1)}} \\ & = \nu^{(i_0, k_{i_0}, NR_{i_0}/k_{i_0})}(\Omega''). \end{aligned}$$

From the above argument and the definition of $\nu^{(i, N, \{c_1, c_2, \dots\})}$ we have that when $N > N_3$,

$$\mathbb{P}^{\mu_r}(A_N \cap \Omega_N) \leq \frac{\sum_{k_1+\dots+k_q=N} Q_{(k_1, \dots, k_q)} \cdot \nu^{(i_0, k_{i_0}, NR_{i_0}/k_{i_0})}(\Omega'')}{\sum_{k_1+\dots+k_q=N} Q_{(k_1, \dots, k_q)}}$$

$$\text{where } \Omega'' = \{ \lambda \in \bar{\Sigma}_{\kappa}^{(k_{i_0})}(i_0) | \exists k \in [\frac{\alpha_*}{2R_{i_0}} k_{i_0}, k_{i_0} - \frac{\alpha_*}{2R_{i_0}} k_{i_0}] \text{ st. } |\lambda_k - \theta(i_0, k_{i_0}, k)| > k_{i_0}^{-1+\frac{1}{2}\epsilon_*} \}.$$

Now we want to use Lemma 3.4. Recall that $a_r = \frac{1}{2}(\frac{3}{4})^{r-1}$, $P_r = P_1 \times 0.2^{r-1}$ and $t_{r+1} = 2t_r + 1.6P_r$.

Now choose r_0 large enough such that $a_{r_0} < \frac{1}{4}\epsilon_*$. Then choose P_1 large enough such that $P_{r_0} > 20$. Then choose $\epsilon_0 \in (0, 0.01)$ small enough such that $t_{r_0}\epsilon_0 < \frac{\epsilon_*}{4}$ and $0.4P_{r_0}\epsilon_0 < \frac{3}{4}a_{r_0}$ for

all $r \in \{1, \dots, r_0 - 1\}$. So $r_0, a_1, \dots, a_{r_0}, t_1, \dots, t_{r_0}, P_1, \dots, P_{r_0}$ and ϵ_0 all depend only on ϵ_* and $a_{r_0} + t_{r_0}\epsilon_0 < \frac{1}{2}\epsilon_*$.

According to Lemma 3.4, for these chosen $r_0, a_1, \dots, a_{r_0}, t_1, \dots, t_{r_0}, P_1, \dots, P_{r_0}$ and ϵ_0 , there exist $C_2 > 0$ and $N_4 > 0$ depending on V, κ, ϵ_* and α_* , such that if $N > N_4$ and $|k_{i_0} - N| \leq N^{\frac{1}{2}\epsilon_*}$, then

$$\begin{aligned} & \nu^{(i_0, k_{i_0}, NR_{i_0}/k_{i_0})}(\Omega'') \\ &= \nu^{(i_0, k_{i_0}, NR_{i_0}/k_{i_0})} \left(\exists k \in \left[\frac{\alpha_*}{2R_{i_0}} k_{i_0}, k_{i_0} - \frac{\alpha_*}{2R_{i_0}} k_{i_0} \right] \text{ st. } |\lambda_k - \theta(i_0, k_{i_0}, k)| > k_{i_0}^{-1+\frac{1}{2}\epsilon_*} \right) \\ &\leq \nu^{(i_0, k_{i_0}, NR_{i_0}/k_{i_0})} \left(\exists k \in \left[\frac{\alpha_*}{2R_{i_0}} k_{i_0}, k_{i_0} - \frac{\alpha_*}{2R_{i_0}} k_{i_0} \right] \text{ st. } |\lambda_k - \theta(i_0, k_{i_0}, k)| > k_{i_0}^{-1+a_{r_0}+t_{r_0}\epsilon_0} \right) \\ &\leq \exp(-k_{i_0}^{C_2}) \leq \exp(-N^{C_2/2}). \end{aligned}$$

So if $N > N_4$, then $\mathbb{P}^{\mu_r}(A_N \cap \Omega_N) \leq \exp(-N^{C_2/2})$. According to Proposition 3.1, there exist $C_3 > 0, N_5 > 0$ depending on V, κ, ϵ_* and α_* such that if $N > N_5$, then $\mathbb{P}^{\mu_\kappa}(A_N \cap \Omega_N) \leq \exp(-N^{2C_3})$ and (because of (4.9))

$$\mathbb{P}^{\mu_\kappa}(A_N) \leq 2 \exp(-N^{C_3}) \quad (4.11)$$

Set

$$E_N = \left\{ \lambda \in \Sigma^{(N)} \mid \exists k \in [(R_1 + \dots + R_{i_0-1} + \alpha_*)N, (R_1 + \dots + R_{i_0} - \alpha_*)N] \text{ st. } |\lambda_k - \eta_k| > N^{-1+\epsilon_*} \right\}.$$

We have

$$\mathbb{P}^{\mu_s}(E_N) \leq \mathbb{P}^{\mu_s}(E_N \cap \Sigma_\kappa^{(N)}) + \mathbb{P}^{\mu_s}(\Sigma^{(N)} \setminus \Sigma_\kappa^{(N)}). \quad (4.12)$$

According to Lemma 2.5, there exist $C_4 > 0, N_6 > 0$ depending on V and κ such that if $N > N_6$, then

$$\mathbb{P}^{\mu_s}(\Sigma^{(N)} \setminus \Sigma_\kappa^{(N)}) < \exp(-N^{C_4}) \quad (4.13)$$

On the other hand,

$$\begin{aligned} \mathbb{P}^{\mu_s}(E_N \cap \Sigma_\kappa^{(N)}) &= \frac{1}{Z_{\mu_s}} \int_{\Sigma_\kappa^{(N)}} \mathbb{1}_{E_N}(\lambda) \exp(-N\beta\mathcal{H}) d\lambda \\ &\leq \frac{1}{Z_{\mu_\kappa}} \int_{\Sigma_\kappa^{(N)}} \mathbb{1}_{A_N}(\lambda) \exp(-N\beta\mathcal{H}) d\lambda \quad (\text{since } Z_{\mu_\kappa} \leq Z_{\mu_s} \text{ and } E_N \cap \Sigma_\kappa^{(N)} = A_N) \\ &= \mathbb{P}^{\mu_\kappa}(A_N). \end{aligned} \quad (4.14)$$

According to (4.11), (4.12), (4.13), (4.14), there are $C_5 > 0, N_7 > 0$ depending on V, κ, ϵ_* and α_* such that if $N > N_7$, then

$$\mathbb{P}^{\mu_s}(E_N) \leq \exp(-N^{C_7}).$$

Since κ depends only on V , the proof of Theorem 1.2 is complete.

5 A β ensemble model in one-cut regime

Consider the β ensemble in one-cut regime and restricted on an interval, i.e., the following probability measure on $[a, b]^N$.

$$\tilde{\mu}(dx) = \tilde{\mu}^{(N)}(dx) = \frac{1}{Z_{\tilde{\mu}}} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N V_p(x_i)} \prod_{i < j} |x_i - x_j|^\beta \prod_{i=1}^N \mathbb{1}_{[a,b]}(x_i) dx$$

where $Z_{\tilde{\mu}}$ is the normalization constant and V_p is the potential.

From this section to the end of this paper we will study $\tilde{\mu}$ and we will always suppose that the following Hypothesis 5.1 is true.

Hypothesis 5.1. *Suppose $\kappa \in (0, 0.1)$, $\epsilon_0 \in (0, 0.01)$ and the following conditions are satisfied.*

1. *The equilibrium measure of $\tilde{\mu}$ is $\tilde{\rho}(x) = \sqrt{(x-c)(d-x)} \tilde{r}(x) \mathbb{1}_{[c,d]}(x)$ where $\tilde{r}(z)$ is analytic on a neighborhood of $\{z \mid \text{dist}(z, [c, d]) \leq 10\kappa\}$. Moreover, $\tilde{r}(z)$ is positive on $[c, d]$ and the distance between the zeros of the complex function $\tilde{r}(z)$ and $[c, d]$ is larger than 10κ . Moreover, $\kappa \in (0, 0.1)$.*
2. *$V_p \in C_0(\mathbb{R})$ and is analytic on $\mathbb{C} \setminus ((-\infty, W_L] \cup [W_R, +\infty))$ where $W_L < c - 100\kappa$, $W_R > d + 100\kappa$.*
3. *For all $x \in \mathbb{R}$, $V_p(x) > 6 \max_{w \in [a,b]} (\int_c^d \tilde{\rho}(y) \ln |w - y| dy) + 4 \ln(b - a + 1) + 2$.*
4. *V_p is off-critical:*

$$\frac{1}{2} V_p(x) - \int_a^b \ln |x - t| \tilde{\rho}(t) dt \begin{cases} = \inf_{x \in [a,b]} \left(\frac{1}{2} V_p(x) - \int_a^b \ln |x - t| \tilde{\rho}(t) dt \right) & \text{if } x \in [c, d] \\ > \inf_{x \in [a,b]} \left(\frac{1}{2} V_p(x) - \int_a^b \ln |x - t| \tilde{\rho}(t) dt \right) & \text{if } x \in [a, b] \setminus [c, d] \end{cases}$$

5. *$[a, b] = [c - \frac{\kappa}{2}, d + \frac{\kappa}{2}]$.*
6. *$|c_N - 1| \leq N^{-1+\epsilon_0}$ for all N .*
7. *There is $U_p > 0$ such that $V_p''(x) \geq -2U_p$ on an open subset of \mathbb{R} which contains $[a, b]$.*

Remark. $\tilde{\mu}$ depends on V_p , κ , ϵ_0 and c_N .

Define $\gamma_k = \gamma_k(N)$ and $\tilde{\gamma}_k = \tilde{\gamma}_k(N)$ by

$$\int_c^{\gamma_k} \tilde{\rho}(t) dt = \frac{k}{N} \quad \int_c^{\tilde{\gamma}_k} \tilde{\rho}(t) dt = \frac{k - 1/2}{N}.$$

Lemma 5.1. *Assume that the Hypothesis 5.1 is satisfied. There exist $\tau_1 > 0$, $\tau_2 > 0$ depending on V_p , κ , ϵ_0 but independent of $\{c_N\}$ such that if $N \geq 1$, then*

$$\begin{aligned} \tau_1 \sqrt{t - c} \leq \tilde{\rho}(t) \leq \tau_2 \sqrt{t - c} & \quad \forall t \in [c, \tilde{\gamma}_{\frac{2}{3}N}], \\ \tau_1 \sqrt{d - t} \leq \tilde{\rho}(t) \leq \tau_2 \sqrt{d - t} & \quad \forall t \in [\tilde{\gamma}_{\frac{1}{3}N}, d]. \end{aligned}$$

We rewrite the density of $\tilde{\mu}$ as

$$\frac{1}{Z_{\tilde{\mu}}} e^{-\beta N \mathcal{H}_{\tilde{\mu}}} \prod_{i=1}^N \mathbb{1}_{[a,b]}(\lambda_i)$$

where $\mathcal{H}_{\tilde{\mu}} = \frac{1}{2} \sum c_N V_p(\lambda_i) - \frac{1}{N} \sum_{i < j} \ln |\lambda_i - \lambda_j|$.

Set $\tilde{\Sigma}_N = \{(x_1, \dots, x_N) | a \leq x_1 < \dots < x_N \leq b\}$. Suppose $\tilde{\mu}_s = \tilde{\mu}_s(N)$ is a probability measures on $\tilde{\Sigma}_N$ with density $\frac{1}{Z_{\tilde{\mu}_s}} \exp(-N\beta \mathcal{H}_{\tilde{\mu}})$ where $Z_{\tilde{\mu}_s}$ is the normalization constants.

Lemma 5.2. *Suppose $f(x_1, \dots, x_N)$ is a symmetric function, then $\mathbb{E}^{\tilde{\mu}}(f(\lambda)) = \mathbb{E}^{\tilde{\mu}_s}(f(\lambda))$. In particular, for $z \notin [a, b]$,*

$$\begin{aligned} \frac{1}{N} \mathbb{E}^{\tilde{\mu}} \sum \frac{1}{z - \lambda_i} &= \frac{1}{N} \mathbb{E}^{\tilde{\mu}_s} \sum \frac{1}{z - \lambda_i}, \\ \mathbb{E}^{\tilde{\mu}} \left[\left(\sum \frac{1}{z - \lambda_i} - \mathbb{E}^{\tilde{\mu}} \sum \frac{1}{z - \lambda_i} \right)^2 \right] &= \mathbb{E}^{\tilde{\mu}_s} \left[\left(\sum \frac{1}{z - \lambda_i} - \mathbb{E}^{\tilde{\mu}_s} \sum \frac{1}{z - \lambda_i} \right)^2 \right] \end{aligned}$$

Proof. Noticing that $\mathcal{H}_{\tilde{\mu}}$ is symmetric in λ and $Z_{\tilde{\mu}} = N! Z_{\tilde{\mu}_s}$, we complete the proof. \square

Suppose $t_1 > P_1 > 0$. For $k \in \{1, 2, \dots\}$, define $a_k = \frac{1}{2}(\frac{3}{4})^{k-1}$, $P_k = P_1 \times 0.2^{k-1}$ and $t_{k+1} = 2t_k + 1.6P_k$.

Theorem 5.3. *Suppose the Hypothesis 5.1 is satisfied. Suppose $\alpha > 0$ and $k \in \mathbb{N}$. If $P_k > 20$ and $0.4P_r \epsilon_0 < \frac{3}{4}a_r$ for all $r \in \{1, \dots, k-1\}$, then there exists $N_0 > 0$ depending on $V_p, \kappa, \epsilon_0, \alpha$ and k such that if $N > N_0$, then*

$$\mathbb{P}^{\tilde{\mu}_s} \left(\exists i \in [\alpha N, (1-\alpha)N] \text{ st. } |\lambda_i - \gamma_i| > N^{-1+a_k+t_k \epsilon_0} \right) \leq \exp(-N^{P_k \epsilon_0}).$$

6 A reference measure: $\tilde{\mu}(h)$

Suppose $C_b > 0$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 function with $\|h\|_{\infty} + \|h'\|_{\infty} + \|h''\|_{\infty} < C_b$.

Consider a probability measure $\tilde{\mu}(h)$ on $[a, b]^N$ with density

$$\begin{aligned} &\frac{1}{Z_{\tilde{\mu}(h)}} \exp\left(-\frac{\beta N}{2} \sum_{i=1}^N c_N V_p(x_i)\right) \exp\left(-\beta \sum_{i=1}^N h(x_i)\right) \prod_{i < j} |x_i - x_j|^{\beta} \prod_{i=1}^N \mathbb{1}_{[a,b]}(x_i) \\ &= \frac{1}{Z_{\tilde{\mu}(h)}} \exp\left(-\frac{\beta N}{2} \sum_{i=1}^N [c_N V_p(x_i) + \frac{2}{N} h(x_i)]\right) \prod_{i < j} |x_i - x_j|^{\beta} \prod_{i=1}^N \mathbb{1}_{[a,b]}(x_i) \\ &= \frac{1}{Z_{\tilde{\mu}(h)}} \exp\left(-\frac{\beta N}{2} \sum_{i=1}^N c_N W_p^N(x_i)\right) \prod_{i < j} |x_i - x_j|^{\beta} \prod_{i=1}^N \mathbb{1}_{[a,b]}(x_i) \end{aligned}$$

where $W_p^N(x) = V_p(x) + \frac{2}{N c_N} h(x)$.

Define $\tilde{\mu}(h)_s$ to be a probability measure on $\tilde{\Sigma}_N$ with density

$$\frac{1}{Z_{\tilde{\mu}(h)_s}} \exp\left(-\frac{\beta N}{2} \sum_{i=1}^N c_N W_p^N(x_i)\right) \prod_{i < j} |x_i - x_j|^\beta \prod_{i=1}^N \mathbb{1}_{[a,b]}(x_i)$$

So if $h \equiv 0$, then $\tilde{\mu}(h) = \tilde{\mu}$ and $\tilde{\mu}(h)_s = \tilde{\mu}_s$. According to Theorem 1.1 of [4], $\tilde{\mu}(h)$ has the same equilibrium measure as $\tilde{\mu}$.

Notice that $(x_1, \dots, x_N) \mapsto \exp\left(-\frac{\beta N}{2} \sum_{i=1}^N c_N W_p^N(x_i)\right) \prod_{i < j} |x_i - x_j|^\beta \prod_{i=1}^N \mathbb{1}_{[a,b]}(x_i)$ is a symmetric function and $Z_{\tilde{\mu}(h)} = N! Z_{\tilde{\mu}(h)_s}$. So similarly as Lemma 5.2 we can prove the following fact.

Lemma 6.1. *Suppose $f(x_1, \dots, x_N)$ is a symmetric function, then $\mathbb{E}^{\tilde{\mu}(h)}(f(\lambda)) = \mathbb{E}^{\tilde{\mu}(h)_s}(f(\lambda))$.*

7 Initial estimation

Theorem 7.1 (Initial estimation). *For any $A_1 > 0$ there exist $A_2 > 0$, $N_1 > 0$ both depending on V_p , κ , ϵ_0 , C_b and A_1 but independent of the choice of $\{c_N\}$ such that if $N > N_1$ then*

$$\mathbb{P}^{\tilde{\mu}(h)_s}(\exists k \in [1, N] \text{ st. } |\lambda_k - \gamma_k| > A_1) \leq \exp(-A_2 N).$$

The purpose of this section is to prove Theorem 7.1.

Recall that $\tilde{\mu}(h)$ is a probability measure on $[a, b]^N$ with following density

$$\frac{1}{Z_{\tilde{\mu}(h)}} e^{-\frac{\beta}{2} \sum_{i=1}^N c_N W_p^N(x_i)} \prod_{i < j} |x_i - x_j|^\beta \prod_{i=1}^N \mathbb{1}_{[a,b]}(x_i).$$

Since $\lambda_1, \dots, \lambda_N$ are Lebesgue-almost surely distinct, we have that the density function is Lebesgue-almost surely equal to

$$\frac{1}{Z_{\tilde{\mu}(h)}} \exp\left(-N^2 \int \int_{x \neq y} f(x, y) dL_N(x) dL_N(y)\right) \prod_{i=1}^N \exp\left(-\frac{\beta}{2} c_N W_p(\lambda_i)\right)$$

where $f(x, y) = \frac{\beta}{4} c_N (W_p^N(x) + W_p^N(y)) - \frac{\beta}{2} \ln |x - y|$ and $L_N(x) = \frac{1}{N} \sum_{i=1}^N \delta(x - \lambda_i)$.

7.1 Large deviation of empirical measure for $\tilde{\mu}(h)$

In this subsection we study the large deviation of empirical measure of $\tilde{\mu}(h)$. We use the method of Section 2.6.1 of [1].

Use $M_1([a, b])$ to denote the space of probability measures on $[a, b]$ equipped with the Lipschitz metric. Therefore $M_1([a, b])$ has a topology induced by the Wasserstein distance $d(\cdot, \cdot)$.

Lemma 7.2. *Suppose $t(x, y)$ is a bounded continuous function on $[a, b]^2$. Then*

$$\nu \mapsto \int t(x, y) d\nu(x) d\nu(y)$$

is a continuous function on $M_1([a, b])$.

Proof. Suppose $\nu_n \rightarrow \nu$. It suffices to prove

$$\int t(x, y) d\nu_n(x) d\nu_n(y) \rightarrow \int t(x, y) d\nu(x) d\nu(y).$$

Use $M_1([a, b]^2)$ to denote the space of probability measures on $[a, b]^2$ equipped with the topology induced by the Wasserstein distance $d(\cdot, \cdot)$. Because of Portmanteau's Theorem (Theorem C.10 of [1]) we only have to show that $\nu_n \otimes \nu_n \rightarrow \nu \otimes \nu$ as elements in $M_1([a, b]^2)$. By definition we have

$$d(\nu_n \otimes \nu_n, \nu \otimes \nu) = \sup_{t \in \Omega} \left| \int t(x, y) d\nu_n(x) d\nu_n(y) - \int t(x, y) d\nu(x) d\nu(y) \right|$$

where Ω is the set of Lipschitz functions from $[a, b]^2$ to $[-1, 1]$ with Lipschitz constant at most 1.

Suppose $t \in \Omega$. Then for any $y \in [a, b]$, $x \mapsto t(x, y)$ is a Lipschitz function from $[a, b]$ to $[-1, 1]$ with Lipschitz constant at most 1. Thus

$$\left| \int t(x, y) d\nu_n(x) - \int t(x, y) d\nu(x) \right| \leq d(\nu_n, \nu)$$

and therefore

$$\begin{aligned} & \left| \int t(x, y) d\nu_n(x) d\nu_n(y) - \int t(x, y) d\nu(x) d\nu(y) \right| \\ & \leq \left| \int t(x, y) d\nu_n(x) d\nu_n(y) - \int t(x, y) d\nu_n(x) d\nu(y) \right| + \left| \int t(x, y) d\nu_n(x) d\nu(y) - \int t(x, y) d\nu(x) d\nu(y) \right| \\ & \leq 2d(\nu_n, \nu). \end{aligned}$$

So $\lim_{n \rightarrow \infty} d(\nu_n \otimes \nu_n, \nu \otimes \nu) \leq \lim_{n \rightarrow \infty} 2d(\nu_n, \nu) = 0$. □

Suppose $I_\beta^{V_p}$ is a function from $M_1([a, b])$ to $[0, +\infty]$ defined by

$$\begin{aligned} I_\beta^{V_p}(\nu) &:= \frac{\beta}{2} \left(\int_a^b V_p(x) d\nu(x) - \int_{[a, b]^2} \ln |x - y| d\nu(x) d\nu(y) \right) \\ &\quad - \frac{\beta}{2} \inf_{\nu \in M_1([a, b])} \left(\int_a^b V_p(x) d\nu(x) - \int_{[a, b]^2} \ln |x - y| d\nu(x) d\nu(y) \right). \end{aligned}$$

Lemma 7.3. $\nu \mapsto I_\beta^{V_p}(\nu)$ is a good rate function. In other words, $I_\beta^{V_p}(\nu) \geq 0$ and

$$\{\nu \in M_1([a, b]) \mid I_\beta^{V_p}(\nu) \leq L\}$$

is compact for all $L \geq 0$.

Proof. According to Prohorov's Theorem, $M_1([a, b])$ is compact.

Suppose $M > 0$. Define

$$\tilde{f}_M(x, y) = \left(\frac{\beta}{4} (V_p(x) + V_p(y)) - \frac{\beta}{2} \ln |x - y| \right) \wedge M.$$

Since \tilde{f}_M is continuous and bounded, Lemma 7.2 tells us that $\nu \mapsto \int \tilde{f}_M(x, y) d\nu(x) d\nu(y)$ is continuous on $M_1([a, b])$. Therefore by monotone convergence theorem

$$\nu \mapsto \frac{\beta}{2} \left(\int_a^b V_p(x) d\nu(x) - \int_{[a, b]^2} \ln |x - y| d\nu(x) d\nu(y) \right) = \sup_{M > 0} \int \tilde{f}_M(x, y) d\nu(x) d\nu(y)$$

is lower semicontinuous. In other words, for any $L \in \mathbb{R}$,

$$\left\{ \nu \in M_1([a, b]) \left| \frac{\beta}{2} \left(\int_a^b V_p(x) d\nu(x) - \int_{[a, b]^2} \ln |x - y| d\nu(x) d\nu(y) \right) \leq L \right. \right\} \quad (7.15)$$

is a closed subset of $M_1([a, b])$. Since $M_1([a, b])$ is compact, (7.15) is compact. Thus for any $L' \in \mathbb{R}$,

$$\left\{ \nu \in M_1([a, b]) \left| I_{\beta}^{V_p}(\nu) \leq L' \right. \right\}$$

is compact.

Obviously $I_{\beta}^{V_p}(\nu)$ is nonnegative. So $I_{\beta}^{V_p}$ is a good rate function. \square

Theorem 7.4. 1. Suppose O is an open subset of $M_1([a, b])$. For any $C > 0$, there exists N_0 depending on $V_p, \epsilon_0, \kappa, C_b, O$ and C such that if $N > N_0$, then

$$\frac{1}{N^2} \ln \mathbb{P}^{\tilde{\mu}(h)}(L_N \in O) \geq - \inf_{\nu \in O} I_{\beta}^{V_p}(\nu) - C.$$

2. Suppose F is a closed subset of $M_1([a, b])$. For any $C > 0$, there exists N_0 depending on $V_p, \epsilon_0, \kappa, C_b, F$ and C such that if $N > N_0$, then

$$\frac{1}{N^2} \ln \mathbb{P}^{\tilde{\mu}(h)}(L_N \in F) \leq - \inf_{\nu \in F} I_{\beta}^{V_p}(\nu) + C.$$

Theorem 7.4 can be proved in the same way as Theorem 2.6.1 of [1].

Corollary 7.5. 1. Suppose O is an open subset of $M_1([a, b])$. For any $C > 0$, there exists N_0 depending on $V_p, \epsilon_0, \kappa, C_b, O$ and C such that if $N > N_0$, then

$$\frac{1}{N^2} \ln \mathbb{P}^{\tilde{\mu}(h)s}(L_N \in O) \geq - \inf_{\nu \in O} I_{\beta}^{V_p}(\nu) - C.$$

2. Suppose F is a closed subset of $M_1([a, b])$. For any $C > 0$, there exists N_0 depending on $V_p, \epsilon_0, \kappa, C_b, F$ and C such that if $N > N_0$, then

$$\frac{1}{N^2} \ln \mathbb{P}^{\tilde{\mu}(h)s}(L_N \in F) \leq - \inf_{\nu \in F} I_{\beta}^{V_p}(\nu) + C.$$

Proof. Notice that $(\lambda_1, \dots, \lambda_N) \mapsto \mathbb{1}_{(\frac{1}{N} \sum \delta(x - \lambda_i) \in O)}$ is a symmetric function. According to Lemma 6.1,

$$\mathbb{P}^{\tilde{\mu}(h)s}(L_N \in O) = \mathbb{E}^{\tilde{\mu}(h)s}(\mathbb{1}_{(\frac{1}{N} \sum \delta(x - \lambda_i) \in O)}) = \mathbb{E}^{\tilde{\mu}(h)}(\mathbb{1}_{(\frac{1}{N} \sum \delta(x - \lambda_i) \in O)}) = \mathbb{P}^{\tilde{\mu}(h)}(L_N \in O).$$

This together with Theorem 7.4 proves the first conclusion. The second conclusion can be proved in the same way. \square

7.2 Initial estimation for extreme eigenvalues λ_{\max} and λ_{\min}

In this subsection we prove the initial estimation for extreme eigenvalues. We use the same method as that in the appendix of [4]. Suppose $\lambda_{\min} = \min\{\lambda_1, \dots, \lambda_N\}$, $\lambda_{\max} = \max\{\lambda_1, \dots, \lambda_N\}$.

Recall that $\tilde{\mu}(h)$ is a probability measure on $[a, b]^N$ with following density

$$\frac{1}{Z_{\tilde{\mu}(h)}} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(x_i)} \prod_{i < j} |x_i - x_j|^\beta \prod_{i=1}^N \mathbb{1}_{[a,b]}(x_i).$$

Define $\mu_{\frac{N-1}{N-1} c_N W_p^N}$ to be the probability measure on $[a, b]^{N-1}$ with density

$$\frac{1}{Z_{\frac{N-1}{N-1} c_N W_p^N}} e^{-\frac{N\beta}{2} \sum_{i=1}^{N-1} c_N W_p^N(x_i)} \prod_{1 \leq i < j \leq N-1} |x_i - x_j|^\beta \prod_{i=1}^{N-1} \mathbb{1}_{[a,b]}(x_i).$$

Suppose F is a closed subset of $[a, c]$. By direct computation we have

$$\mathbb{P}^{\tilde{\mu}(h)}(\lambda_{\min} \in F) = Y_N \int_{z \in F} e^{-\frac{1}{2} \beta c_N W_p^N(z)} \Xi_N(z) dz \quad (7.16)$$

where

$$Y_N = \frac{Z_{\frac{N-1}{N-1} c_N W_p^N}}{Z_{\tilde{\mu}(h)}} \quad \text{and}$$

$$\Xi_N(z) = \mathbb{E}^{\mu_{\frac{N-1}{N-1} c_N W_p^N}} \left(e^{-\frac{1}{2} \beta c_N (N-1) W_p^N(z)} \prod_{i=1}^{N-1} \mathbb{1}_{[a, \lambda_i]}(z) |\lambda_i - z|^\beta \right)$$

7.2.1 Estimation of $\Xi_N(z)$

Set $L_{N-1}(x) = \frac{1}{N-1} \sum_{i=1}^{N-1} \delta(x - \lambda_i)$.

For any $\tau > 0$,

$$\begin{aligned} \Xi_N(z) &= \mathbb{E}^{\mu_{\frac{N-1}{N-1} c_N W_p^N}} \left(e^{-\frac{1}{2} \beta c_N (N-1) W_p^N(z)} \left(\prod_{i=1}^{N-1} \mathbb{1}_{[a, \lambda_i]}(z) |\lambda_i - z|^\beta \right) \mathbb{1}_{d(L_{N-1}, \tilde{\rho}(t)) < \tau} \right) \\ &\quad + \mathbb{E}^{\mu_{\frac{N-1}{N-1} c_N W_p^N}} \left(e^{-\frac{1}{2} \beta c_N (N-1) W_p^N(z)} \left(\prod_{i=1}^{N-1} \mathbb{1}_{[a, \lambda_i]}(z) |\lambda_i - z|^\beta \right) \mathbb{1}_{d(L_{N-1}, \tilde{\rho}(t)) \geq \tau} \right) \end{aligned}$$

$$:= I + II$$

where $d(\cdot, \cdot)$ is the Wasserstein distance.

By direct computation we have

$$\begin{aligned} I &= \mathbb{E}^{\mu_{\frac{N-1}{N-1} c_N W_p^N}} \left(e^{\beta(N-1)(-\frac{1}{2} V_p(z) + \int \ln |z-x| dL_{N-1}(x)) + \beta(N-1)(-\frac{1}{2} V_p(z))(c_N-1) - \frac{N-1}{N} \beta h(z)} \left(\prod_{i=1}^{N-1} \mathbb{1}_{[a, \lambda_i]}(z) \right) \mathbb{1}_{d(L_N, \tilde{\rho}(t)) < \tau} \right) \\ &\leq \exp \left(\beta(N-1) \sup_{d(\nu, \tilde{\rho}(t)) < \tau} \left(-\frac{1}{2} V_p(z) + \int \ln |z-x| d\nu(x) \right) \right) \cdot \exp \left(\frac{1}{2} \beta \left(\sup_{x \in [a,b]} |V_p(x)| \right) N^{\epsilon_0} + \beta C_b \right) \end{aligned}$$

Lemma 7.6.

$$\limsup_{\tau \downarrow 0} \sup_{z \in F} \sup_{d(\nu, \tilde{\rho}(t)dt) < \tau} \left(-\frac{1}{2}V_p(z) + \int \ln |z - x| d\nu(x) \right) \leq -\inf_{z \in F} \left[\frac{1}{2}V_p(z) - \int \ln |z - x| \tilde{\rho}(x) dx \right]$$

Proof. For $z \in F$ and $0 < \xi < b - a$, define $g_{z,\xi}$ to be a function on $[a, b]$ by

$$g_{z,\xi}(x) = \ln(\max(|z - x|, \xi)).$$

It is easy to check that

$$\|g_{z,\xi}\|_{\mathcal{L}} \leq \frac{1}{\xi}, \quad \|g_{z,\xi}\|_{\infty} \leq \max(|\ln \xi|, |\ln(b - a)|).$$

Set $w(\xi) = \max(\frac{1}{\xi}, |\ln \xi|, |\ln(b - a)|)$, then $\frac{1}{w(\xi)}g_{z,\xi}(x)$ is a Lipschitz function from $[a, b]$ to $[-1, 1]$ with Lipschitz constant no more than 1. Thus

$$\begin{aligned} \left| \int g_{z,\xi}(x) d\nu(x) - \int g_{z,\xi}(x) \tilde{\rho}(x) dx \right| &= w(\xi) \left| \int \frac{1}{w(\xi)} g_{z,\xi}(x) d\nu(x) - \int \frac{1}{w(\xi)} g_{z,\xi}(x) \tilde{\rho}(x) dx \right| \\ &\leq w(\xi) d(\nu, \tilde{\rho}(t)dt). \end{aligned}$$

Therefore

$$\begin{aligned} \sup_{z \in F} \sup_{d(\nu, \tilde{\rho}(t)dt) < \tau} \left(-\frac{1}{2}V_p(z) + \int \ln |z - x| d\nu(x) \right) &\leq \sup_{z \in F} \sup_{d(\nu, \tilde{\rho}(t)dt) < \tau} \left(-\frac{1}{2}V_p(z) + \int g_{z,\xi}(x) d\nu(x) \right) \\ &\leq \sup_{z \in F} \left(-\frac{1}{2}V_p(z) + \int g_{z,\xi}(x) \tilde{\rho}(x) dx \right) + w(\xi)\tau. \end{aligned}$$

Since F is closed and contained in $[a, c]$, $\xi_0 := \frac{1}{2}(c - \sup_{x \in F} x)$ must be positive. For any $z \in F$ and $x \in \text{supp}(\tilde{\rho}(t)dt) = [c, d]$, we have $|z - x| > \xi_0$ and thus $g_{z,\xi_0}(x) = \ln |z - x|$. Therefore

$$\sup_{z \in F} \sup_{d(\nu, \tilde{\rho}(t)dt) < \tau} \left(-\frac{1}{2}V_p(z) + \int \ln |z - x| d\nu(x) \right) \leq \sup_{z \in F} \left(-\frac{1}{2}V_p(z) + \int \ln |z - x| \tilde{\rho}(x) dx \right) + w(\xi_0)\tau.$$

For any $\epsilon > 0$, we can choose τ small enough such that $w(\xi_0)\tau < \epsilon$. Then

$$\limsup_{\tau \rightarrow 0} \sup_{z \in F} \sup_{d(\nu, \tilde{\rho}(t)dt) < \tau} \left(-\frac{1}{2}V_p(z) + \int \ln |z - x| d\nu(x) \right) \leq \sup_{z \in F} \left(-\frac{1}{2}V_p(z) + \int \ln |z - x| \tilde{\rho}(x) dx \right) + \epsilon.$$

Since ϵ is arbitrary, we complete the proof. \square

For any $C_0 > 0$, choose τ small enough such that

$$\sup_{z \in F} \sup_{d(\nu, \tilde{\rho}(t)dt) < \tau} \left(-\frac{1}{2}V_p(z) + \int \ln |z - x| d\nu(x) \right) \leq -\inf_{z \in F} \left[\frac{1}{2}V_p(z) - \int \ln |z - x| \tilde{\rho}(x) dx \right] + C_0$$

Then for all $z \in F$ and $N \in \mathbb{N}$,

$$\begin{aligned} I &\leq \exp \left(-\beta(N-1) \inf_{z \in F} \left(\frac{1}{2}V_p(z) - \int \ln |z - x| \tilde{\rho}(x) dx \right) \right) \\ &\quad \cdot \exp \left(\beta(N-1)C_0 + \frac{1}{2}\beta \left(\sup_{x \in [a,b]} |V_p(x)| \right) N^{\epsilon_0} + \beta C_b \right) \end{aligned} \tag{7.17}$$

For this chosen τ , according to Theorem 7.4, there are $C_1 > 0$ and $N_2 > 0$ depending on V_p , ϵ_0 , κ , C_b and τ such that if $N > N_2$ then

$$\mathbb{P}^{\mu_{N-1}^{\frac{N}{N-1}c_N W_p^N}}(d(\tilde{\rho}(t)dt, L_{N-1}) \geq \tau) \leq e^{-C_1 N^2}.$$

So when $N > N_2$ we have

$$II \leq e^{\frac{1}{2}\beta c_N(N-1) \sup_{x \in [a,b]} |V_p(x)| + \beta C_b} \cdot |b-a|^{\beta(N-1)} \cdot e^{-C_1 N^2} \quad (7.18)$$

From (7.17) and (7.18) we have:

Lemma 7.7. *For any $C_2 > 0$, there are $N_3 > 0$ depending on V_p , ϵ_0 , κ , C_b and C_2 such that for all $z \in F$ and $N > N_3$ we have*

$$\Xi_N(z) \leq \exp\left(C_2(N-1) - \beta(N-1) \inf_{y \in F} \left(\frac{1}{2}V_p(y) - \int \ln|y-x|\tilde{\rho}(x)dx\right)\right)$$

7.2.2 Estimation of Y_N

By direct computation,

$$\frac{1}{Y_N} = \mathbb{E}^{\mu_{N-1}^{\frac{N}{N-1}c_N W_p^N}} \left[\int_a^b e^{-\frac{1}{2}N\beta c_N W_p^N(t)} \prod_{i=1}^{N-1} |\lambda_i - t|^\beta dt \right].$$

Suppose $\epsilon > 0$ and $x \in [a + \epsilon, b - \epsilon]$. Then

$$\frac{1}{Y_N} \geq \mathbb{E}^{\mu_{N-1}^{\frac{N}{N-1}c_N W_p^N}} \left[\int_{x-\epsilon}^{x+\epsilon} e^{-\frac{1}{2}N\beta c_N W_p^N(t)} \prod_{i=1}^{N-1} |\lambda_i - t|^\beta dt \right].$$

Notice that

$$\begin{aligned} & \int_{x-\epsilon}^{x+\epsilon} e^{-\frac{1}{2}N\beta c_N W_p^N(t)} \prod_{i=1}^{N-1} |\lambda_i - t|^\beta dt \\ & \geq \exp\left(-\frac{1}{2}N\beta V_p(x) - \frac{1}{2}N\beta|c_N - 1| \sup_{t \in [a,b]} |V_p(t)| - \frac{1}{2}N\beta \sup_{|t-x| \leq \epsilon} |V_p(x) - V_p(t)| - \beta C_b\right) \int_{x-\epsilon}^{x+\epsilon} e^{\beta \sum_{i=1}^{N-1} \ln|\lambda_i - t|} dt \\ & \geq \exp\left(-\frac{1}{2}N\beta V_p(x) - \frac{1}{2}N\beta|c_N - 1| \sup_{t \in [a,b]} |V_p(t)| - \frac{1}{2}N\beta \sup_{|t-x| \leq \epsilon} |V_p(x) - V_p(t)| - \beta C_b\right) 2\epsilon e^{\frac{\beta}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} \sum_{i=1}^{N-1} \ln|\lambda_i - t| dt} \\ & \geq \exp\left(-\frac{1}{2}N\beta V_p(x) - \frac{1}{2}N^{\epsilon_0}\beta \sup_{t \in [a,b]} |V_p(t)| - \frac{1}{2}N\beta \sup_{|t-x| \leq \epsilon} |V_p(x) - V_p(t)| - \beta C_b\right) 2\epsilon e^{\frac{\beta}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} \sum_{i=1}^{N-1} \ln|\lambda_i - t| dt} \end{aligned}$$

The second inequality comes from Jensen's inequality. Set

$$\delta(\epsilon) = \max(\epsilon, \beta \sup_{\substack{|t_1 - t_2| \leq \epsilon \\ t_1, t_2 \in [a,b]}} |V_p(t_1) - V_p(t_2)|).$$

It is easy to see $\lim_{\epsilon \downarrow 0} \delta(\epsilon) = 0$.

There is $N_4 > 0$ depending on V_p , ϵ_0 , κ , C_b and ϵ such that for all $N > N_4$ and $x \in [a + \epsilon, b - \epsilon]$ we have

$$\frac{1}{2} N^{\epsilon_0} \beta \sup_{t \in [a, b]} |V_p(t)| + \frac{1}{2} N \beta \sup_{|t-x| \leq \epsilon} |V_p(x) - V_p(t)| + \beta C_b \leq N \delta(\epsilon)$$

and thus

$$\frac{1}{Y_N} \geq 2\epsilon \exp\left(-\frac{1}{2} N \beta V_p(x) - N \delta(\epsilon)\right) \mathbb{E}^{\mu_{\frac{N-1}{N-1}} c_N W_p^N} \left[e^{\frac{\beta}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} \sum_{i=1}^{N-1} \ln |\lambda_i - t| dt} \right] \quad (7.19)$$

Suppose $\epsilon > 0$ and $x \in [a + \epsilon, b - \epsilon]$. For $w \in [a, b]$ define $p(x, w) = \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} \ln |w - t| dt$.

Lemma 7.8. 1. Suppose $\epsilon > 0$. There exists $D_0 > 0$ depending only on $[a, b]$ such that for any $x \in [a + \epsilon, b - \epsilon]$ and $w \in [a, b]$, we have

$$|p(x, w)| \leq D_0 \ln \epsilon.$$

2. Suppose $\epsilon > 0$. For every $x \in [a + \epsilon, b - \epsilon]$ we have: $w \mapsto p(x, w)$ is bounded and continuous on $[a, b]$.

3. $\lim_{\epsilon \downarrow 0} \int_c^d \tilde{\rho}(t) p(c, t) dt = \int_c^d \tilde{\rho}(t) \ln |c - t| dt$.

Proof. By direct computation we have

$$p(x, w) = -1 + \frac{1}{2\epsilon} \left[(x + \epsilon - w) \ln |x + \epsilon - w| - (x - \epsilon - w) \ln |x - \epsilon - w| \right], \quad \forall x \in [a + \epsilon, b - \epsilon], w \in [a, b].$$

The first two results comes immediately from the above expression of $p(x, w)$.

Now we prove the third result. By direct computation,

$$\begin{aligned} \int_c^d \tilde{\rho}(t) [\ln |c - t| - p(c, t)] dt &= \int_c^{c+\sqrt{\epsilon}} \tilde{\rho}(t) [\ln |c - t| - p(c, t)] dt + \int_{c+\sqrt{\epsilon}}^d \tilde{\rho}(t) [\ln |c - t| - p(c, t)] dt \\ &:= I + II \end{aligned} \quad (7.20)$$

• Estimation of I :

$$\begin{aligned} |I| &\leq \int_c^{c+\sqrt{\epsilon}} \|\tilde{\rho}\|_{\infty} [|\ln |c - t|| + D_0 \ln \epsilon] dt \leq \|\tilde{\rho}\|_{\infty} D_0 (\ln \epsilon) \sqrt{\epsilon} + \|\tilde{\rho}\|_{\infty} \int_0^{\sqrt{\epsilon}} |\ln x| dx \\ &= \|\tilde{\rho}\|_{\infty} D_0 (\ln \epsilon) \sqrt{\epsilon} + \|\tilde{\rho}\|_{\infty} \sqrt{\epsilon} (1 - \ln(\sqrt{\epsilon})). \end{aligned} \quad (7.21)$$

- Estimation of II : By definition of $p(c, t)$,

$$\begin{aligned}
II &= - \int_{c+\sqrt{\epsilon}}^d \tilde{\rho}(t) \frac{1}{2\epsilon} [(c-t+\epsilon) \ln |\frac{c-t+\epsilon}{c-t}| - (c-t-\epsilon) \ln |\frac{c-t-\epsilon}{c-t}| - 2\epsilon] dt \\
&= - \int_{c+\sqrt{\epsilon}}^d \tilde{\rho}(t) \frac{1}{2\epsilon} \left[(c-t+\epsilon) \sum_{k=1}^{\infty} \left(\frac{(-1)^{k-1}}{k} \left(\frac{\epsilon}{c-t} \right)^k \right) - (c-t-\epsilon) \sum_{k=1}^{\infty} \left(\frac{(-1)^{k-1}}{k} \left(\frac{-\epsilon}{c-t} \right)^k \right) - 2\epsilon \right] dt \\
&= - \int_{c+\sqrt{\epsilon}}^d \tilde{\rho}(t) \frac{1}{2\epsilon} (c-t) \left[\frac{2}{3} \left(\frac{\epsilon}{c-t} \right)^3 + \frac{2}{5} \left(\frac{\epsilon}{c-t} \right)^5 + \frac{2}{7} \left(\frac{\epsilon}{c-t} \right)^7 + \dots \right] dt \\
&\quad - \int_{c+\sqrt{\epsilon}}^d \tilde{\rho}(t) \left[-\frac{1}{2} \left(\frac{\epsilon}{c-t} \right)^2 - \frac{1}{4} \left(\frac{\epsilon}{c-t} \right)^4 - \frac{1}{6} \left(\frac{\epsilon}{c-t} \right)^6 - \dots \right] dt
\end{aligned}$$

Thus

$$\begin{aligned}
|II| &\leq \int_{c+\sqrt{\epsilon}}^d \tilde{\rho}(t) \left[\frac{1}{3} \left(\frac{\epsilon}{c-t} \right)^4 + \frac{1}{5} \left(\frac{\epsilon}{c-t} \right)^6 + \frac{1}{7} \left(\frac{\epsilon}{c-t} \right)^8 + \dots \right] dt \\
&\quad + \int_{c+\sqrt{\epsilon}}^d \tilde{\rho}(t) \left[\frac{1}{2} \left(\frac{\epsilon}{c-t} \right)^2 + \frac{1}{4} \left(\frac{\epsilon}{c-t} \right)^4 + \frac{1}{6} \left(\frac{\epsilon}{c-t} \right)^6 + \dots \right] dt \\
&\leq \left[\frac{1}{3} (\sqrt{\epsilon})^4 + \frac{1}{5} (\sqrt{\epsilon})^6 + \frac{1}{7} (\sqrt{\epsilon})^8 + \dots \right] + \left[\frac{1}{2} (\sqrt{\epsilon})^2 + \frac{1}{4} (\sqrt{\epsilon})^4 + \frac{1}{6} (\sqrt{\epsilon})^6 + \dots \right] \\
&\leq 2 \left[(\sqrt{\epsilon})^2 + (\sqrt{\epsilon})^4 + (\sqrt{\epsilon})^6 + \dots \right] \\
&= \frac{2\epsilon}{1-\epsilon} \\
&\leq 4\epsilon.
\end{aligned} \tag{7.22}$$

(Here we used the fact that $|\frac{-\epsilon}{c-t}| \leq \sqrt{\epsilon}$ for $t \in [c+\sqrt{\epsilon}, d]$.)

(7.20), (7.21) and (7.22) complete the proof of the third result. \square

Lemma 7.9. *For any $C_3 > 0$, there exists $N_6 > 0$ depending on V_p , ϵ_0 , κ , C_b and C_3 such that if $N > N_6$, then*

$$\frac{1}{Y_N} \geq e^{-C_3 N - N\beta \inf_{r \in [a, b]} \left(\frac{1}{2} V_p(r) - \int_a^b \ln |r-t| \tilde{\rho}(t) dt \right)}$$

Proof. Recall that $\lim_{\epsilon \downarrow 0} \delta(\epsilon) = 0$. This together with Lemma 7.8 implies that there exists $\epsilon < 1$ (depending on C_3 , V_p , ϵ_0 and κ) such that $\delta(\epsilon) < 0.1C_3$ and

$$\left| \int_c^d \tilde{\rho}(t) p(c, t) dt - \int_c^d \tilde{\rho}(t) \ln |c-t| dt \right| < 0.1C_3.$$

Now fix this ϵ . According to the second result of Lemma 7.8, $w \mapsto p(c, w)$ is bounded and continuous on $[a, b]$. By Portmanteau's Theorem (see, for example, Theorem C.10 of [1]), there exists $r > 0$ depending on V_p , κ , ϵ_0 and ϵ such that if $\pi \in M_1([a, b])$ and $d(\pi, \tilde{\rho}(t) dt) \leq r$, then

$$\left| \int_a^b p(c, w) d\pi(w) - \int_a^b p(c, w) \tilde{\rho}(t) dw \right| < 0.1C_3.$$

According to (7.19), there is $N_4 > 0$ depending on V_p , ϵ_0 , κ , C_b and ϵ such that for all $N > N_4$,

$$\begin{aligned}
\frac{1}{Y_N} &\geq 2\epsilon \exp\left(-\frac{1}{2}N\beta V_p(c) - N\delta(\epsilon)\right) \mathbb{E}^{\mu_{N-1}^{\frac{N}{N-1}c_N W_p^N}} \left[e^{\frac{\beta}{2\epsilon} \int_{c-\epsilon}^{c+\epsilon} \sum_{i=1}^{N-1} \ln |\lambda_i - t| dt} \right] \\
&= 2\epsilon \exp\left(-\frac{1}{2}N\beta V_p(c) - N\delta(\epsilon)\right) \mathbb{E}^{\mu_{N-1}^{\frac{N}{N-1}c_N W_p^N}} \left[e^{\beta(N-1) \int_a^b p(c,w) dL_{N-1}(w)} \right] \\
&\geq 2\epsilon \exp\left(-\frac{1}{2}N\beta V_p(c) - 0.1C_3N\right) \mathbb{E}^{\mu_{N-1}^{\frac{N}{N-1}c_N W_p^N}} \left[e^{\beta(N-1) \int_a^b p(c,w) dL_{N-1}(w)} \mathbf{1}_{d(L_{N-1}, \tilde{\rho}(t)dt) \leq r} \right] \\
&\geq 2\epsilon \exp\left(-\frac{1}{2}N\beta V_p(c) - 0.1C_3N\right) e^{\beta(N-1)(-0.1C_3 + \int_a^b p(c,w) \tilde{\rho}(w) dw)} \\
&\quad \times \mathbb{P}^{\mu_{N-1}^{\frac{N}{N-1}c_N W_p^N}}(d(L_{N-1}, \tilde{\rho}(t)dt) \leq r) \\
&\geq 2\epsilon \exp\left(-\frac{1}{2}N\beta V_p(c) - 0.1C_3N\right) e^{\beta(N-1)(-0.2C_3 + \int_a^b \ln |c-w| \tilde{\rho}(w) dw)} \\
&\quad \times \mathbb{P}^{\mu_{N-1}^{\frac{N}{N-1}c_N W_p^N}}(d(L_{N-1}, \tilde{\rho}(t)dt) \leq r)
\end{aligned}$$

Using Theorem 7.4 with N replaced by $N-1$ and c_{N-1} replaced by $\frac{c_N N}{N-1}$ (thus $\tilde{\mu}(h)$ becomes $\mu_{N-1}^{\frac{N}{N-1}c_N W_p^N}$) we see that there is $N_5 > 0$ depending on V_p , ϵ_0 , κ , C_b and ϵ such that when $N > N_5$ we have

$$\mathbb{P}^{\mu_{N-1}^{\frac{N}{N-1}c_N W_p^N}}(d(L_{N-1}, \tilde{\rho}(t)dt) \leq r) > \frac{1}{2}.$$

Thus when $N > N_5(V_p, \epsilon_0, \kappa, C_b, \epsilon)$ we have

$$\begin{aligned}
\frac{1}{Y_N} &\geq \epsilon \exp\left(-\frac{1}{2}N\beta V_p(c) - 0.1C_3N\right) e^{\beta(N-1)(-0.2C_3 + \int_a^b \ln |c-w| \tilde{\rho}(w) dw)} \\
&= \epsilon e^{N\beta(-\frac{1}{2}V_p(c) + \int_a^b \ln |c-t| \tilde{\rho}(t) dt) - 0.1C_3N - 0.2C_3\beta(N-1) - \beta \int_a^b \tilde{\rho}(t) \ln |c-t| dt} \\
&= \epsilon e^{-N\beta \inf_{r \in [a,b]} \left(\frac{1}{2}V_p(r) - \int_a^b \ln |r-t| \tilde{\rho}(t) dt\right) - 0.1C_3N - 0.2C_3\beta(N-1) - \beta \int_a^b \tilde{\rho}(t) \ln |c-t| dt} \quad (7.23)
\end{aligned}$$

(The last step comes from Condition 4 of the Hypothesis 5.1.)

Since C_3 is arbitrary, the proof is complete. \square

7.2.3 Conclusion: initial estimation for λ_{\max} and λ_{\min}

Recall Condition 4 of the Hypothesis 5.1. By (7.16), Lemma 7.7 and Lemma 7.9 we have the following theorem.

Theorem 7.10. *For any $\tau > 0$, there are $C_4 > 0$, $N_7 > 0$ depending on V_p , ϵ_0 , κ , C_b and τ such that if $N > N_7$ then*

$$\mathbb{P}^{\tilde{\mu}(h)}(\lambda_{\min} \leq c - \tau) \leq e^{-C_4N}, \quad \mathbb{P}^{\tilde{\mu}(h)}(\lambda_{\max} \geq d + \tau) \leq e^{-C_4N}.$$

Remark. The result for λ_{\max} can be proved in the same way as that for λ_{\min} .

Corollary 7.11. For any $\tau > 0$, there are $C_4 > 0$, $N_7 > 0$ depending on V_p , ϵ_0 , κ , C_b and τ such that if $N > N_7$ then

$$\mathbb{P}^{\tilde{\mu}(h)s}(\lambda_1 \leq c - \tau) \leq e^{-C_4 N}, \quad \mathbb{P}^{\tilde{\mu}(h)s}(\lambda_N \geq d + \tau) \leq e^{-C_4 N}.$$

Proof. Notice that $(\lambda_1, \dots, \lambda_N) \mapsto 1 - \prod_{i=1}^N \mathbb{1}_{(\lambda_i > c - \tau)}$ is a symmetric function. According to Lemma 6.1,

$$\begin{aligned} \mathbb{P}^{\tilde{\mu}(h)s}(\lambda_1 \leq c - \tau) &\leq e^{-C_4 N} = \mathbb{E}^{\tilde{\mu}(h)s}(\mathbb{1}_{(\lambda_1 \leq c - \tau)}) = \mathbb{E}^{\tilde{\mu}(h)s}(1 - \mathbb{1}_{(\lambda_1 > c - \tau)}) = \mathbb{E}^{\tilde{\mu}(h)s}\left(1 - \prod_{i=1}^N \mathbb{1}_{(\lambda_i > c - \tau)}\right) \\ &= \mathbb{E}^{\tilde{\mu}(h)}\left(1 - \prod_{i=1}^N \mathbb{1}_{(\lambda_i > c - \tau)}\right) = \mathbb{E}^{\tilde{\mu}(h)}(\mathbb{1}_{(\lambda_{\min} \leq c - \tau)}) = \mathbb{P}^{\tilde{\mu}(h)}(\lambda_{\min} \leq c - \tau). \end{aligned}$$

This together with Theorem 7.10 proves the first conclusion. The second conclusion can be proved in the same way. \square

7.3 Initial estimation for all eigenvalues

In this section we prove Theorem 7.1. First we recall some notations we defined before.

L_N denotes the empirical measure: $L_N = \frac{1}{N} \sum_{i=1}^N \delta(x - \lambda_i)$.

$M_1([a, b])$ denotes the set of probability measures on $[a, b]$ with topology induced by the Lipschitz metric:

$$d_{LU}(\mu_1, \mu_2) = \sup_{f \in \mathcal{F}_{LU}} \left| \int_a^b f(x) d\mu_1(x) - \int_a^b f(x) d\mu_2(x) \right|$$

where \mathcal{F}_{LU} is the set of Lipschitz functions $f : [a, b] \rightarrow [-1, 1]$ with Lipschitz constant at most 1.

$I_\beta^{V_p}$ is a function from $M_1([a, b])$ to $[0, +\infty]$ defined by

$$\begin{aligned} I_\beta^{V_p}(\nu) &:= \frac{\beta}{2} \left(\int_a^b V_p(x) d\nu(x) - \int_{[a, b]^2} \ln |x - y| d\nu(x) d\nu(y) \right) \\ &\quad - \frac{\beta}{2} \inf_{\nu \in M_1([a, b])} \left(\int_a^b V_p(x) d\nu(x) - \int_{[a, b]^2} \ln |x - y| d\nu(x) d\nu(y) \right). \end{aligned}$$

For any $a > 0$, set $F_a = \{\nu \in M_1([a, b]) | d_{LU}(\nu, \tilde{\rho}(t) dt) \geq a\}$.

Lemma 7.12. For any $a > 0$ we have

$$\inf_{\nu \in F_a} I_\beta^{V_p}(\nu) > 0.$$

Proof. By definition

$$\begin{aligned} \inf_{\nu \in F_a} I_\beta^{V_p}(\nu) &= \frac{\beta}{4} \inf_{\nu \in F_a} \iint \left[V_p(x) + V_p(y) - 2 \ln |x - y| \right] d\nu(x) d\nu(y) \\ &\quad - \frac{\beta}{4} \inf_{\nu \in M_1([a, b])} \iint \left[V_p(x) + V_p(y) - 2 \ln |x - y| \right] d\nu(x) d\nu(y) \end{aligned}$$

By Theorem 1.1 of [4], $\tilde{\rho}(t)dt$ is the only minimizer (in $M_1([a, b])$) of

$$\nu \mapsto \iint \left[V_p(x) + V_p(y) - 2 \ln |x - y| \right] d\nu(x) d\nu(y).$$

Therefore we only need to show that there exists $\nu_0 \in F_a$ such that

$$\inf_{\nu \in F_a} \iint \left[V_p(x) + V_p(y) - 2 \ln |x - y| \right] d\nu(x) d\nu(y) = \iint \left[V_p(x) + V_p(y) - 2 \ln |x - y| \right] d\nu_0(x) d\nu_0(y)$$

or equivalently,

$$I_\beta^{V_p}(\nu_0) = \inf_{\nu \in F_a} I_\beta^{V_p}(\nu).$$

Set

$$A = \{\nu \in F_a \mid I_\beta^{V_p}(\nu) \leq 1 + \inf_{\nu \in F_a} I_\beta^{V_p}(\nu)\} \quad \text{and} \quad A_k = \{\nu \in F_a \mid I_\beta^{V_p}(\nu) \leq \frac{1}{k} + \inf_{\nu \in F_a} I_\beta^{V_p}(\nu)\} \quad (k = 1, 2, \dots).$$

Then A is a closed subset of $\{\nu \in M_1([a, b]) \mid I_\beta^{V_p}(\nu) \leq 1 + \inf_{\nu \in F_a} I_\beta^{V_p}(\nu)\}$ and A_k is a closed subset of $\{\nu \in M_1([a, b]) \mid I_\beta^{V_p}(\nu) \leq \frac{1}{k} + \inf_{\nu \in F_a} I_\beta^{V_p}(\nu)\}$. By Lemma 7.3, A and every A_k are compact and closed.

If $\cap_{k=1}^\infty A_k = \emptyset$, then $\{A \setminus A_1, A \setminus A_2, \dots\}$ is a class of open subsets of A which covers A . Since A is compact, there exists $k_0 \in \mathbb{N}$ such that $\cup_{k=1}^{k_0} A \setminus A_k = A$. So $\cap_{k=1}^{k_0} A_k = \emptyset$ or equivalently, $A_{k_0} = \{\nu \in F_a \mid I_\beta^{V_p}(\nu) \leq \frac{1}{k_0} + \inf_{\nu \in F_a} I_\beta^{V_p}(\nu)\} = \emptyset$ which is a contradiction.

Therefore we proved that $\cap_{k=1}^\infty A_k \neq \emptyset$. It is easy to check that if $\nu_0 \in \cap_{k=1}^\infty A_k$, then

$$I_\beta^{V_p}(\nu_0) = \inf_{\nu \in F_a} I_\beta^{V_p}(\nu).$$

□

Lemma 7.13. Suppose $\lambda = (\lambda_1, \dots, \lambda_N) \in \tilde{\Sigma}_N$. For any $\alpha > 0$, $\epsilon > 0$ and $1 \leq i \leq q$ there exist $N_0 > 0$ and $\tilde{\epsilon} > 0$ depending on V_p , κ , α , ϵ such that if $N > N_0$ and

$$\exists k \in [\alpha N, (1 - \alpha)N] \quad \text{with} \quad |\lambda_k - \gamma_k| > \epsilon,$$

then $d_{LU}(L_N, \tilde{\rho}(t)dt) \geq \tilde{\epsilon}$.

Proof. Recall that $\tilde{\rho}(t) > 0$ on the interior of its support. So there must be $c > 0$ and $\delta > 0$ depending on V_p , κ , α and ϵ such that

1. $\delta \leq \epsilon$,
2. if $k \in [\alpha N, (1 - \alpha)N]$ and $t \in [\gamma_k - \delta, \gamma_k + \delta]$, then $\tilde{\rho}(t) > c$.

Fix a natural number $N_0 > \frac{4}{c\delta}$. Suppose $N > N_0$ and $|\lambda_k - \gamma_k| > \epsilon$ where $k \in [\alpha N, (1 - \alpha)N]$.

If $\lambda_k < \gamma_k - \epsilon$, set

$$f(x) = \begin{cases} 0 & \text{if } x \leq \gamma_k - \delta \\ x - (\gamma_k - \delta) & \text{if } x \in [\gamma_k - \delta, \gamma_k] \\ \delta & \text{if } x \geq \gamma_k \end{cases}$$

It is easy to check that $f \in \mathcal{F}_{LU}$. We have

$$\int f(x)\tilde{\rho}(x)dx = \int_{\gamma_k-\delta}^{\gamma_k} (x - (\gamma_k - \delta))\tilde{\rho}(x)dx + \delta \int_{\gamma_k}^{+\infty} \tilde{\rho}(x)dx \geq \frac{1}{2}c\delta^2 + \delta(1 - \frac{k}{N})$$

and

$$\int f(x)L_N(x)dx = \frac{1}{N} \sum f(\lambda_i) \leq \frac{N+1-k}{N}\delta.$$

Thus $\int f(x)\tilde{\rho}(x)dx - \int f(x)L_N(x)dx \geq \frac{1}{2}c\delta^2 + \frac{\delta}{N} > \frac{1}{4}c\delta^2$ and $d_{LU}(L_N, \tilde{\rho}(t)dt) > \frac{1}{4}c\delta^2$.

If $\lambda_k > \gamma_k + \epsilon$, set

$$g(x) = \begin{cases} 0 & \text{if } x \geq \gamma_k + \delta \\ (\gamma_k + \delta) - x & \text{if } x \in [\gamma_k, \gamma_k + \delta] \\ \delta & \text{if } x \leq \gamma_k \end{cases}$$

Similarly we can prove that $g \in \mathcal{F}_{LU}$ and $\int g(x)\tilde{\rho}(x)dx - \int g(x)L_N(x)dx > \frac{1}{4}c\delta^2$. So we still have $d_{LU}(L_N, \tilde{\rho}(t)dt) > \frac{1}{4}c\delta^2$.

Setting $\tilde{\epsilon} = \frac{1}{4}c\delta^2$ we complete the proof. \square

Proof of Theorem 7.1. Suppose

$$0 < \alpha < \min(\int_c^{c+\frac{1}{2}A_1} \tilde{\rho}(t)dt, \int_{d-\frac{1}{2}A_1}^d \tilde{\rho}(t)dt).$$

Thus if $i \in [1, \alpha N]$, then $\gamma_i \in (c, c + \frac{1}{2}A_1)$; if $i \in [(1-\alpha)N, N]$, then $\gamma_i \in (d - \frac{1}{2}A_1, d]$.

Suppose $\lambda \in \tilde{\Sigma}_N$ and there exists $k \in [1, N]$ such that $|\lambda_k - \gamma_k| > A_1$. Thus one of the following cases must happen.

1. $k \in [1, \alpha N]$ and $\lambda_k < \gamma_k - A_1$. In this case, $\lambda_1 \leq \lambda_k < \gamma_k - A_1 < c - \frac{1}{2}A_1$.
2. $k \in [1, \alpha N]$ and $\lambda_k > \gamma_k + A_1$. In this case, $\lambda_{\alpha N} \geq \lambda_k > \gamma_k + A_1 > c + A_1 > \gamma_{\alpha N} + \frac{1}{2}A_1$.
3. $k \in [(1-\alpha)N, N]$ and $\lambda_k > \gamma_k + A_1$. In this case, $\lambda_N \geq \lambda_k > \gamma_k + A_1 > d - \frac{1}{2}A_1 + A_1 = d + \frac{1}{2}A_1$.
4. $k \in [(1-\alpha)N, N]$ and $\lambda_k < \gamma_k - A_1$. In this case, $\lambda_{(1-\alpha)N} \leq \lambda_k < \gamma_k - A_1 \leq d - A_1 < \gamma_{(1-\alpha)N} - \frac{1}{2}A_1$.
5. $k \in [\alpha N, (1-\alpha)N]$ and $|\lambda_k - \gamma_k| > A_1$.

In summary, $\lambda_1 < c - \frac{1}{2}A_1$ or $\lambda_N > d + \frac{1}{2}A_1$ or there exists $k \in [\alpha N, (1-\alpha)N]$ such that $|\lambda_k - \gamma_k| > \frac{1}{2}A_1$. Therefore we have

$$\begin{aligned} \mathbb{P}^{\tilde{\mu}^{(h)}_s}(\exists k \in [1, N] \text{ st. } |\lambda_k - \gamma_k| > A_1) &\leq \mathbb{P}^{\tilde{\mu}^{(h)}_s}(\lambda_1 < c - \frac{1}{2}A_1) + \mathbb{P}^{\tilde{\mu}^{(h)}_s}(\lambda_N > d + \frac{1}{2}A_1) \\ &\quad + \mathbb{P}^{\tilde{\mu}^{(h)}_s}(\exists k \in [\alpha N, (1-\alpha)N] \text{ st. } |\lambda_k - \gamma_k| > \frac{1}{2}A_1). \end{aligned}$$

Using Corollary 7.11, Lemma 7.13, Corollary 7.5 and Lemma 7.12 we complete the proof. \square

8 Analysis of the loop equation

Now we derive the loop equation of $\tilde{\mu}(h)$ (which was defined in Section 6). We use the method of change-of-variable developed by Johansson [16]. For $z \in \mathbb{C} \setminus [c, d]$. Use $\tilde{m}(z)$ to denote the Stieltjes transform of the equilibrium measure $\tilde{\rho}(t)dt$:

$$\tilde{m}(z) = \int_c^d \frac{1}{z-t} \tilde{\rho}(t) dt.$$

Recall that $\tilde{\rho}(x) = \sqrt{(x-c)(d-x)} \tilde{r}(x) \mathbf{1}_{[c,d]}(x)$ where $\tilde{r}(z)$ is analytic on a neighborhood of $\{z \mid \text{dist}(z, [c, d]) \leq 10\kappa\}$. Moreover, $\tilde{r}(z)$ is positive on $[c, d]$ and the distance between the zeros of the complex function $\tilde{r}(z)$ and $[c, d]$ is at least 10κ . (See the Hypothesis 5.1.)

Lemma 8.1. *Suppose $z \in \{z \mid 0 < \text{dist}(z, [c, d]) < 6\kappa\}$. Then*

$$2\tilde{m}(z) - V'_p(z) = -2\tilde{r}(z)\sqrt{(c-z)(d-z)}$$

where the square root is defined such that $\sqrt{(c-z)(d-z)} \sim z$ as $z \rightarrow \infty$.

This lemma is in fact a Riemann-Hilbert problem. We leave the proof of it in Appendix B.

Suppose $\phi(x) : \mathbb{R} \rightarrow \mathbb{C}$ is C^∞ with $\|\phi\|_\infty < +\infty$ and $\|\phi'\|_\infty < +\infty$. Set $x_i = y_i + \lambda\phi(y_i)$ where $|\lambda|$ is small enough such that $\lambda\phi'(x) > -1$ for all $x \in \mathbb{R}$. Then we have

$$\begin{aligned} Z_{\tilde{\mu}(h)} &= \int_{[a,b]^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(x_i)} \prod_{i < j} |x_i - x_j|^\beta dx \\ &= \int_{[a',b']^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i + \lambda\phi(y_i))} \prod_{i < j} |y_i - y_j + \lambda(\phi(y_i) - \phi(y_j))|^\beta \prod_{i=1}^N (1 + \lambda\phi'(y_i)) dy \end{aligned}$$

where $a = a' + \lambda\phi(a')$, $b = b' + \lambda\phi(b')$. We have $\lim_{\lambda \rightarrow 0} a' = a$ and $\lim_{\lambda \rightarrow 0} b' = b$ because $\|\phi\|_\infty < +\infty$. Therefore

$$\begin{aligned} 0 &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left(\int_{[a',b']^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i + \lambda\phi(y_i))} \prod_{i < j} |y_i - y_j + \lambda(\phi(y_i) - \phi(y_j))|^\beta \prod_{i=1}^N (1 + \lambda\phi'(y_i)) dy \right. \\ &\quad \left. - \int_{[a,b]^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i)} \prod_{i < j} |y_i - y_j|^\beta dy \right) \end{aligned}$$

and

$$\begin{aligned}
& \int_{[a', b']^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i + \lambda\phi(y_i))} \prod_{i < j} |y_i - y_j + \lambda(\phi(y_i) - \phi(y_j))|^\beta \prod_{i=1}^N (1 + \lambda\phi'(y_i)) dy \\
& - \int_{[a, b]^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i)} \prod_{i < j} |y_i - y_j|^\beta dy \\
& = \left[\int_{[a, b]^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i + \lambda\phi(y_i))} \prod_{i < j} |y_i - y_j + \lambda(\phi(y_i) - \phi(y_j))|^\beta \prod_{i=1}^N (1 + \lambda\phi'(y_i)) \right. \\
& \quad \left. - e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i)} \prod_{i < j} |y_i - y_j|^\beta dy \right] \\
& + \left[\int_{[a', b']^N \setminus [a, b]^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i + \lambda\phi(y_i))} \prod_{i < j} |y_i - y_j + \lambda(\phi(y_i) - \phi(y_j))|^\beta \prod_{i=1}^N (1 + \lambda\phi'(y_i)) dy \right] \\
& - \left[\int_{[a, b]^N \setminus [a', b']^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i + \lambda\phi(y_i))} \prod_{i < j} |y_i - y_j + \lambda(\phi(y_i) - \phi(y_j))|^\beta \prod_{i=1}^N (1 + \lambda\phi'(y_i)) dy \right] \\
& := I + II - III.
\end{aligned}$$

Set $A_i^N = \{(y_1, \dots, y_N) | y_i \in [a', b'] \setminus [a, b], y_j \in [a', b'] \text{ for all } j \neq i\}$. Then

$$\begin{aligned}
|II| & \leq \sum_{i=1}^N \int_{A_i^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i + \lambda\phi(y_i))} \prod_{i < j} |y_i - y_j + \lambda(\phi(y_i) - \phi(y_j))|^\beta \prod_{i=1}^N (1 + \lambda\phi'(y_i)) dy \\
& \leq N \int_{[a', b'] \setminus [a, b]} e^{-\frac{N\beta}{2} c_N W_p^N(y + \lambda\phi(y))} D^{(N-1)\beta} (1 + |\lambda| \|\phi'\|_\infty) dy \\
& \quad \cdot \int_{[a', b']^{N-1}} e^{-\frac{N\beta}{2} \sum_{i=1}^{N-1} c_N W_p^N(y_i + \lambda\phi(y_i))} \prod_{1 \leq i < j \leq N-1} |y_i - y_j + \lambda(\phi(y_i) - \phi(y_j))|^\beta \prod_{i=1}^{N-1} (1 + \lambda\phi'(y_i)) dy
\end{aligned}$$

where $D = \sup\{|u - v + \lambda(\phi(u) - \phi(v))| | u \in [a', b'] \setminus [a, b], v \in [a', b']\}$.

Now we estimate $|\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \frac{II}{Z_{\tilde{\mu}(h)}}|$. First, for any $\delta_1 > 0$, if $|\lambda|$ is small enough then $[a', b'] \setminus [a, b] \subset [a - \delta_1, a + \delta_1] \cup [b - \delta_1, b + \delta_1]$, thus

$$\begin{aligned}
& \left| \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{[a', b'] \setminus [a, b]} e^{-\frac{N\beta}{2} c_N W_p^N(y + \lambda\phi(y))} D^{(N-1)\beta} (1 + |\lambda| \|\phi'\|_\infty) dy \right| \\
& \leq \left| \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} |[a', b'] \setminus [a, b]| e^{-\frac{N\beta}{2} c_N \inf_{x \in [a - \delta_1, a + \delta_1] \cup [b - \delta_1, b + \delta_1]} W_p^N(x)} D^{(N-1)\beta} (1 + |\lambda| \|\phi'\|_\infty) \right| \\
& \leq 2 \|\phi\|_\infty \cdot e^{-\frac{N\beta}{2} c_N \inf_{x \in [a - \delta_1, a + \delta_1] \cup [b - \delta_1, b + \delta_1]} W_p^N(x)} (b - a + 1)^{(N-1)\beta} \\
& \leq 2 \|\phi\|_\infty \cdot e^{-\frac{N\beta}{2} c_N \min(W_p^N(a), W_p^N(b))} (b - a + 1)^{(N-1)\beta}
\end{aligned}$$

The last inequality comes from the fact that δ_1 is arbitrarily small. Second,

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \frac{1}{Z_{\tilde{\mu}(h)}} \int_{[a', b']^{N-1}} e^{-\frac{N\beta}{2} \sum_{i=1}^{N-1} c_N W_p^N(y_i + \lambda \phi(y_i))} \prod_{1 \leq i < j \leq N-1} |y_i - y_j + \lambda(\phi(y_i) - \phi(y_j))|^\beta \prod_{i=1}^{N-1} (1 + \lambda \phi'(y_i)) dy \\ &= \frac{1}{Z_{\tilde{\mu}(h)}} \int_{[a, b]^{N-1}} e^{-\frac{N\beta}{2} \sum_{i=1}^{N-1} c_N W_p^N(y_i)} \prod_{1 \leq i < j \leq N-1} |y_i - y_j|^\beta dy \end{aligned}$$

which is the same as Y_N defined in Section 7.2. According to Lemma 7.9, there exists $N_0 > 0$ depending on V_p , ϵ_0 , κ and C_b such that if $N > N_0$, then

$$Y_N \leq e^{\frac{1}{2}N\beta \inf_{r \in [a, b]} \left(\frac{1}{2}V_p(r) - \int_a^b \ln |r-t| \tilde{\rho}(t) dt \right)}$$

Therefore when $N > N_0(V_p, \epsilon_0, \kappa, C_b)$,

$$\begin{aligned} & \left| \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \frac{II}{Z_{\tilde{\mu}(h)}} \right| \\ & \leq 2 \|\phi\|_\infty \cdot e^{-\frac{N\beta}{2} c_N \min(W_p^N(a), W_p^N(b))} (b-a+1)^{(N-1)\beta} \\ & \quad \times \exp \left(\frac{1}{2} \beta N \inf_{x \in [a, b]} \left[\frac{V_p(x)}{2} - \int_c^d \tilde{\rho}(y) \ln |x-y| dy \right] \right) \\ & = 2 \|\phi\|_\infty \cdot e^{-\frac{N\beta}{2} c_N \min(W_p^N(a), W_p^N(b))} (b-a+1)^{(N-1)\beta} \\ & \quad \times \exp \left(\frac{1}{2} \beta N \left[\frac{V_p(c)}{2} - \int_c^d \tilde{\rho}(y) \ln |c-y| dy \right] \right) \quad (\text{Condition 4 of Hypothesis 5.1}) \end{aligned}$$

Notice

$$\begin{aligned} e^{-\frac{N\beta}{2} c_N \min(W_p^N(a), W_p^N(b))} &= e^{-\frac{N\beta}{2} c_N \min(V_p(a) + \frac{2}{Nc_N} h(a), V_p(b) + \frac{2}{Nc_N} h(b))} \\ &\leq e^{-\frac{N\beta}{2} c_N \min(V_p(a), V_p(b)) + \beta \|h\|_\infty} \\ &\leq e^{-\frac{N\beta}{2} \min(V_p(a), V_p(b)) + \frac{N\beta}{2} |c_N - 1| \sup_{t \in [a, b]} |V_p(t)| + \beta \|h\|_\infty} \\ &\leq e^{-\frac{N\beta}{2} \min(V_p(a), V_p(b)) + \frac{\beta}{2} N^{\epsilon_0} \sup_{t \in [a, b]} |V_p(t)| + \beta \|h\|_\infty}. \end{aligned}$$

For convenience we use $H(x)$ to denote $V_p(x) - 2 \int_c^d \tilde{\rho}(y) \ln |x - y| dy$. Therefore

$$\begin{aligned}
& \left| \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \frac{II}{Z_{\tilde{\mu}(h)}} \right| \\
& \leq 2 \|\phi\|_{\infty} \cdot e^{-\frac{N\beta}{2} \min(V_p(a), V_p(b)) + \frac{\beta}{2} N^{\epsilon_0} \sup_{t \in [a, b]} |V_p(t)| + \beta \|h\|_{\infty}} \cdot e^{(N-1)\beta \ln(b-a+1)} \cdot \exp\left(\frac{1}{4} \beta N H(c)\right) \\
& \leq 2 \|\phi\|_{\infty} \cdot e^{-\frac{N\beta}{2} \min(H(a), H(b)) + N\beta \max_{x \in [a, b]} (\int_c^d \tilde{\rho}(y) \ln |x-y| dy) + \frac{\beta}{2} N^{\epsilon_0} \sup_{t \in [a, b]} |V_p(t)| + \beta \|h\|_{\infty}} \\
& \quad \cdot e^{(N-1)\beta \ln(b-a+1)} \cdot \exp\left(\frac{1}{4} \beta N H(c)\right) \\
& \leq 2 \|\phi\|_{\infty} \cdot e^{N\beta \max_{x \in [a, b]} (\int_c^d \tilde{\rho}(y) \ln |x-y| dy) + \frac{\beta}{2} N^{\epsilon_0} \sup_{t \in [a, b]} |V_p(t)| + \beta \|h\|_{\infty} + (N-1)\beta \ln(b-a+1)} \\
& \quad \times e^{-\frac{N\beta}{4} H(c)} \quad (\text{since } \min(H(a), H(b)) \geq H(c) \text{ by Condition 4 of Hypothesis 5.1})
\end{aligned}$$

According to Condition 3 of Hypothesis 5.1,

$$H(c) > 4 \max_{x \in [a, b]} \left(\int_c^d \tilde{\rho}(y) \ln |x - y| dy \right) + 4 \ln(b - a + 1) + 2.$$

Thus for $N > N_8(V_p, \epsilon_0, \kappa, C_b)$,

$$\left| \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \frac{II}{Z_{\tilde{\mu}(h)}} \right| \leq 2 \|\phi\|_{\infty} e^{-\frac{N\beta}{4}}$$

Similarly, for $N > N_8$,

$$\left| \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \frac{III}{Z_{\tilde{\mu}(h)}} \right| \leq 2 \|\phi\|_{\infty} \cdot \exp\left(-\frac{N\beta}{4}\right).$$

Finally,

$$\begin{aligned}
& \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \frac{I}{Z_{\tilde{\mu}(h)}} \\
& = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \frac{1}{Z_{\tilde{\mu}(h)}^N} \left[\int_{[a, b]^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i + \lambda \phi(y_i))} \prod_{i < j} |y_i - y_j + \lambda(\phi(y_i) - \phi(y_j))|^{\beta} \prod_{i=1}^N (1 + \lambda \phi'(y_i)) \right. \\
& \quad \left. - e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i)} \prod_{i < j} |y_i - y_j|^{\beta} dy \right] \\
& = \frac{1}{Z_{\tilde{\mu}(h)}} \int_{[a, b]^N} \frac{\partial}{\partial \lambda} \Big|_{\lambda=0} \left[e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i + \lambda \phi(y_i))} \prod_{i < j} |y_i - y_j + \lambda(\phi(y_i) - \phi(y_j))|^{\beta} \prod_{i=1}^N (1 + \lambda \phi'(y_i)) \right] dy \\
& = \frac{1}{Z_{\tilde{\mu}(h)}} \int_{[a, b]^N} e^{-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^N(y_i)} \prod_{i < j} |y_i - y_j|^{\beta} \left[-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^{N'}(y_i) \phi(y_i) + \sum_{i < j} \beta \frac{\phi(y_i) - \phi(y_j)}{y_i - y_j} + \sum_{i=1}^N \phi'(y_i) \right] dy \\
& = \mathbb{E}_{\tilde{\mu}(h)} \left[-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^{N'}(y_i) \phi(y_i) + \sum_{i < j} \beta \frac{\phi(y_i) - \phi(y_j)}{y_i - y_j} + \sum_{i=1}^N \phi'(y_i) \right]
\end{aligned}$$

We summarize the above argument to be the following Proposition.

Proposition 8.1. *Suppose $\phi(x) : \mathbb{R} \rightarrow \mathbb{C}$ is C^∞ with $\|\phi\|_\infty < +\infty$ and $\|\phi'\|_\infty < +\infty$. There exists $N_0 > 0$ depending on ϵ_0 , V_p , κ and C_b but independent of the choice of $\{c_N\}$ such that if $N > N_0$, then*

$$\left| \mathbb{E}_{\tilde{\mu}(h)} \left[-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^{N'}(y_i) \phi(y_i) + \sum_{i < j} \beta \frac{\phi(y_i) - \phi(y_j)}{y_i - y_j} + \sum_{i=1}^N \phi'(y_i) \right] \right| \leq 4\|\phi\|_\infty \cdot \exp\left(-\frac{N\beta}{4}\right). \quad (8.24)$$

Remark. According to Lemma 6.1, (8.24) is also true if we replace $\tilde{\mu}(h)$ by $\tilde{\mu}(h)_s$.

9 Fluctuation of linear statistics of eigenvalues under $\tilde{\mu}$

In this section we estimate the fluctuation of linear statistics of eigenvalues under $\tilde{\mu}$. The main result of this section is the following theorem. It is an analogue of Lemma 6.5 of [8].

Theorem 9.1. *Suppose $h : \mathbb{R} \rightarrow \mathbb{R}$ is C^2 and $\|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty < C_b$. Suppose $\tau > \epsilon_0$ and $C_0 \in (0, \tau)$. There exists $N_0 > 0$ depending on V_p , ϵ_0 , κ , C_b , C_0 and τ such that if $N > N_0$, then*

$$\mathbb{P}^{\tilde{\mu}} \left(\left| \sum_{i=1}^N h(x_i) - N \int h(x) \tilde{\rho}(x) dx \right| > N^\tau \right) < \exp(-N^{C_0}).$$

Corollary 9.2. *Suppose $h : \mathbb{R} \rightarrow \mathbb{R}$ is C^2 and $\|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty < C_b$. Suppose $\tau > \epsilon_0$ and $C_0 \in (0, \tau)$. There exists $N_0 > 0$ depending on V_p , ϵ_0 , κ , C_b , C_0 and τ such that if $N > N_0$, then*

$$\mathbb{P}^{\tilde{\mu}_s} \left(\left| \sum_{i=1}^N h(x_i) - N \int h(x) \tilde{\rho}(x) dx \right| > N^\tau \right) < \exp(-N^{C_0}).$$

Proof of Corollary 9.2. Notice that $(x_1, \dots, x_N) \mapsto \sum_{i=1}^N h(x_i) - N \int h(x) \tilde{\rho}(x) dx$ is a symmetric function. According to Lemma 5.2,

$$\begin{aligned} \mathbb{P}^{\tilde{\mu}_s} \left(\left| \sum_{i=1}^N h(x_i) - N \int h(x) \tilde{\rho}(x) dx \right| > N^\tau \right) &= \mathbb{E}^{\tilde{\mu}_s} \left(\mathbb{1}_{\left(\left| \sum_{i=1}^N h(x_i) - N \int h(x) \tilde{\rho}(x) dx \right| > N^\tau \right)} \right) \\ &= \mathbb{E}^{\tilde{\mu}} \left(\mathbb{1}_{\left(\left| \sum_{i=1}^N h(x_i) - N \int h(x) \tilde{\rho}(x) dx \right| > N^\tau \right)} \right) = \mathbb{P}^{\tilde{\mu}} \left(\left| \sum_{i=1}^N h(x_i) - N \int h(x) \tilde{\rho}(x) dx \right| > N^\tau \right). \end{aligned}$$

Then the statement is induced from Theorem 9.1. □

9.1 Preliminaries

Recall that $\tilde{\mu}(h)$ is a probability measure on $[a, b]^N$ with density

$$\begin{aligned} & \frac{1}{Z_{\tilde{\mu}(h)}} \exp\left(-\frac{\beta N}{2} \sum_{i=1}^N c_N V_p(x_i)\right) \exp\left(-\beta \sum_{i=1}^N h(x_i)\right) \prod_{i < j} |x_i - x_j|^\beta \prod_{i=1}^N \mathbb{1}_{[a, b]}(x_i) \\ &= \frac{1}{Z_{\tilde{\mu}(h)}} \exp\left(-\frac{\beta N}{2} \sum_{i=1}^N [c_N V_p(x_i) + \frac{2}{N} h(x_i)]\right) \prod_{i < j} |x_i - x_j|^\beta \prod_{i=1}^N \mathbb{1}_{[a, b]}(x_i) \end{aligned}$$

Suppose $\tilde{\rho}_1^{(h, N)}(x)$ and $\tilde{\rho}_2^{(h, N)}(x, y)$ are the one-point and two-points correlation functions of $\tilde{\mu}(h)$ respectively. Set

$$\begin{aligned} \tilde{m}(z) &= \int \frac{1}{z-t} \tilde{\rho}(t) dt, \quad \tilde{m}_{h, N}(z) = \int \frac{1}{z-t} \tilde{\rho}_1^{(h, N)}(t) dt = \frac{1}{N} \sum \mathbb{E}^{\tilde{\mu}(h)} \frac{1}{z - \lambda_i} \\ \text{Var}_{\tilde{\mu}(h)}\left(\sum \frac{1}{z - \lambda_i}\right) &= \mathbb{E}^{\tilde{\mu}(h)} \left[\left(\sum \frac{1}{z - \lambda_i}\right)^2\right] - \left(\mathbb{E}^{\tilde{\mu}(h)} \sum \frac{1}{z - \lambda_i}\right)^2. \end{aligned}$$

For $\xi \in \mathbb{C}$, $d(\xi)$ denotes the distance between ξ and $[a, b]$.

Recall that h is C^2 and $\|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty < C_b$.

From Theorem 1.1 of [4], $\tilde{\mu}(h)$ has the same equilibrium measure as $\tilde{\mu}$.

Lemma 9.3. *Suppose $\phi : \mathbb{R} \rightarrow \mathbb{C}$ is bounded and differentiable. There exists $N_0 > 0$, $C_1 > 0$ satisfying the following conditions.*

1. N_0 and C_1 depend on V_p , κ , ϵ_0 and C_b and are independent of $\{c_N\}$, h and ϕ .
2. If $N > N_0$, then $|\int \phi(x)(\tilde{\rho}(x) - \tilde{\rho}_1^{(h, N)}(x))dx| \leq C_1 \|\phi'\|_2^{1/2} \|\phi\|_2^{1/2} \sqrt{\frac{\ln N}{N}}$.
3. If $N > N_0$, then $|\int \phi(x)\phi(y)(\tilde{\rho}_1^{(h, N)}(x)\tilde{\rho}_1^{(h, N)}(y) - \tilde{\rho}_2^{(h, N)}(x, y))dxdy| \leq C_1 \|\phi'\|_2 \|\phi\|_2 \frac{\ln N}{N}$.

Lemma 9.3 can be proved in the same way as Theorem 2.1 (ii) of [18].

Lemma 9.4. *There exist $C_2 > 0$, $N_0 > 0$ depending on V_p , ϵ_0 , κ and C_b such that if $N > N_0$ and $d(\xi) > 0$, then*

1. $|\tilde{m}_{h, N}(\xi) - \tilde{m}(\xi)| \leq C_2 \frac{1}{d(\xi)} \sqrt{\frac{\ln N}{N}}$;
2. $|\frac{1}{N^2} \text{Var}_{\tilde{\mu}(h)}(\sum \frac{1}{\xi - \lambda_k})| \leq C_2 \frac{1}{d(\xi)^2} \frac{\ln N}{N}$.

Proof. Notice that $\tilde{m}_{h, N}(\xi) - \tilde{m}(\xi) = \int \frac{1}{\xi - t} (\tilde{\rho}_1^{(h, N)}(t) - \tilde{\rho}(t)) dt$ and

$$\begin{aligned} \frac{1}{N^2} \text{Var}_{\tilde{\mu}(h)}\left(\sum \frac{1}{\xi - \lambda_k}\right) &= \frac{N-1}{N} \int \frac{1}{\xi - t} \frac{1}{\xi - s} \left[\tilde{\rho}_2^{(h, N)}(s, t) - \tilde{\rho}_1^{(h, N)}(s) \tilde{\rho}_1^{(h, N)}(t) \right] ds dt \\ &\quad - \frac{1}{N} \left(\int \frac{1}{\xi - t} \tilde{\rho}_1^{(h, N)}(t) dt \right)^2 + \frac{1}{N} \int \frac{1}{(\xi - t)^2} \tilde{\rho}_1^{(h, N)}(t) dt. \end{aligned}$$

If we can find a constant C_3 depending only on V_p , κ and ϵ_0 such that for each $z \in \{z | d(z) > 0\}$ there exists $f_z : \mathbb{R} \rightarrow \mathbb{C}$ satisfying

1. $f_z(t) = \frac{1}{z-t}$ if $t \in [a, b]$
2. $\|f_z\|_\infty \leq \frac{C_3}{d(z)}$
3. $\|f_z\|_2^{1/2} \|f'_z\|_2^{1/2} \leq \frac{C_3}{d(z)}$

then the proof is complete by Lemma 9.3. Now we construct such f_z .

Suppose $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function with $\phi(x) = 0$ when $x \leq 0$, $\phi(x) = 1$ when $x \geq 1$ and $\|\phi\|_\infty = 1$. Suppose $p_1(x) = 3x^3 + 5x^2 + x$ and $p_2(x) = -x^3 + x^2 + x$. Set

$$f_z(x) = \begin{cases} 0 & \text{if } x \leq a - 2|z - a| \\ \frac{1}{z-a} \left(1 + \frac{\overline{z-a}}{|z-a|}\right) \phi\left(\frac{x-a}{|z-a|} + 2\right) & \text{if } a - 2|z-a| \leq x \leq a - |z-a| \\ \frac{1}{z-a} \frac{\overline{z-a}}{|z-a|} p_1\left(\frac{x-a}{|z-a|}\right) + \frac{1}{z-a} & \text{if } a - |z-a| \leq x \leq a \\ \frac{1}{z-x} & \text{if } a \leq x \leq b \\ \frac{1}{z-b} \frac{\overline{z-b}}{|z-b|} p_2\left(\frac{x-b}{|z-b|}\right) + \frac{1}{z-b} & \text{if } b \leq x \leq b + |z-b| \\ \frac{1}{z-b} \left(1 + \frac{\overline{z-b}}{|z-b|}\right) \phi\left(2 - \frac{x-b}{|z-b|}\right) & \text{if } b + |z-b| \leq x \leq b + 2|z-b| \\ 0 & \text{if } x \geq b + 2|z-b| \end{cases}$$

By direct computation it can be checked that for every $z \in \{z | d(z) > 0\}$, we have $f_z \in C^1(\mathbb{R})$, $\|f_z\|_\infty \leq \frac{3}{d(z)}$ and $\|f_z\|_2^{1/2} \|f'_z\|_2^{1/2} \leq \frac{1}{d(z)} (72 + 26\|\phi'\|_\infty + \sqrt{\frac{13}{d-c+\kappa}})$. \square

Lemma 9.5. *There exists $C_4 > 0$, $N_0 > 0$ depending on V_p , κ , ϵ_0 and C_b such that if $N > N_0$ and $z = E + i\eta \in \{E + i\eta | E \in \mathbb{R}, \eta \in (0, 1]\}$ we have*

$$\begin{aligned} |(z-c)(z-d)|^{1/2} |\tilde{m}_{h,N}(z) - \tilde{m}(z)| &\leq C_4 \frac{\sqrt{\ln N}}{N\eta^2} |(z-c)(z-d)|^{1/2} \quad \text{if } \eta \in (0, N^{-1/2}]; \\ |(z-c)(z-d)|^{1/2} |\tilde{m}_{h,N}(z) - \tilde{m}(z)| &\leq C_4 \sqrt{\ln N} |(z-c)(z-d)|^{1/2} \quad \text{if } \eta \in [N^{-1/2}, 1]. \end{aligned}$$

Remark. Lemma 9.5 can be directly induced from Lemma 9.4.

By Proposition 8.1 we have: when $N > N_0(\epsilon_0, V_p, \kappa, C_b)$,

$$\begin{aligned} &\left| \mathbb{E}_{\tilde{\mu}(h)} \left[-\frac{N\beta}{2} \sum_{i=1}^N c_N W_p^{N'}(y_i) \phi(y_i) + \sum_{i < j} \beta \frac{\phi(y_i) - \phi(y_j)}{y_i - y_j} + \sum_{i=1}^N \phi'(y_i) \right] \right| \\ &= \left| -\frac{N^2\beta}{2} c_N \int_a^b W_p^{N'}(t) \phi(t) \tilde{\rho}_1^{(h,N)}(t) dt + N \int_a^b \phi'(t) \tilde{\rho}_1^{(h,N)}(t) dt + \frac{\beta}{2} N(N-1) \int_{[a,b]^2} \frac{\phi(t) - \phi(s)}{t-s} \tilde{\rho}_2^{(h,N)}(t,s) dt ds \right| \\ &\leq 4\|\phi\|_\infty \cdot \exp\left(-\frac{N\beta}{4}\right). \end{aligned}$$

Set $\phi(t) = \frac{1}{z-t}$ with $z \in \mathbb{C} \setminus ([a, b] \cup (-\infty, W_L] \cup [W_R, +\infty))$. (W_L and W_R are defined in the Hypothesis 5.1.) By direct computation,

$$\begin{aligned} & \left| (\tilde{m}_{h,N}(z))^2 - c_N V_p'(z) \tilde{m}_{h,N}(z) + \int_a^b \frac{c_N V_p'(z) - c_N W_p^{N'}(t)}{z-t} \tilde{\rho}_1^{(h,N)}(t) dt + \frac{1}{N^2} \text{Var}_{\tilde{\mu}(h)} \left(\sum \frac{1}{z-\lambda_i} \right) - \frac{1}{N} \left(\frac{2}{\beta} - 1 \right) \tilde{m}'_{h,N}(z) \right| \\ & \leq \frac{2}{N^2 \beta} \cdot \frac{4}{\text{dist}(z, [a, b])} \cdot \exp \left(-\frac{N\beta}{4} \right) \leq \frac{2}{N^2 \beta} \cdot \frac{4}{\eta} \cdot \exp \left(-\frac{N\beta}{4} \right) \end{aligned} \quad (9.25)$$

and $\eta = \text{Im} z$. Obviously $\|\phi\|_\infty \leq \frac{1}{\text{dist}(z, [a, b])} \leq \frac{1}{\eta}$.

Because of Lemma 9.4, $\tilde{m}_{h,N}(z) \rightarrow \tilde{m}(z)$, $\tilde{m}'_{h,N}(z) \rightarrow \tilde{m}'(z)$ and $\frac{1}{N^2} \text{Var}_{\tilde{\mu}(h)} \left(\sum \frac{1}{z-\lambda_i} \right) \rightarrow 0$ as $N \rightarrow \infty$. Let $N \rightarrow \infty$ for both sides of (9.25), then

$$\tilde{m}^2(z) - V_p'(z) \tilde{m}(z) + \int_c^d \frac{V_p'(z) - V_p'(t)}{z-t} \tilde{\rho}(t) dt = 0. \quad (9.26)$$

From (9.25) and (9.26) there exists $N_0 > 0$ depending on ϵ_0 , V_p , κ and C_b , but independent of the choice of $\{c_N\}$ such that if $N > N_0$ and $z \in \mathbb{C} \setminus ([a, b] \cup (-\infty, W_L] \cup [W_R, +\infty))$, then

$$\begin{aligned} & \left| (\tilde{m}_{h,N}(z) - \tilde{m}(z))^2 + (2\tilde{m}(z) - V_p'(z))(\tilde{m}_{h,N}(z) - \tilde{m}(z)) + \int_a^b \frac{V_p'(z) - V_p'(t)}{z-t} (\tilde{\rho}_1^{(h,N)}(t) - \tilde{\rho}(t)) dt \right. \\ & + \int_a^b \frac{V_p'(t) - W_p^{N'}(t)}{z-t} \tilde{\rho}_1^{(h,N)}(t) dt + (1 - c_N) V_p'(z) \tilde{m}_{h,N}(z) + (c_N - 1) \int_a^b \frac{V_p'(z) - W_p^{N'}(t)}{z-t} \tilde{\rho}_1^{(h,N)}(t) dt \\ & \left. + \frac{1}{N^2} \text{Var}_{\tilde{\mu}(h)} \left(\sum \frac{1}{z-\lambda_i} \right) - \frac{1}{N} \left(\frac{2}{\beta} - 1 \right) \tilde{m}'_{h,N}(z) \right| \\ & \leq \frac{2}{N^2 \beta} \cdot \frac{4}{\text{dist}(z, [a, b])} \cdot \exp \left(-\frac{N\beta}{4} \right) \leq \frac{2}{N^2 \beta} \cdot \frac{4}{\eta} \cdot \exp \left(-\frac{N\beta}{4} \right). \end{aligned} \quad (9.27)$$

9.2 Induction

In this section we always have $z = E + i\eta$. For $k \in \{1, 2, \dots\}$, define $a_k = (\frac{1}{2})^k$. For $k \in \{1, 2, \dots\}$ set $\mathcal{P}(k)$ to be the following statement.

There exist $C_{ind} = C_{ind}(V_p, \kappa, \epsilon_0, C_b) > 0$ and $N_0 = N_0(V_p, \kappa, \epsilon_0, C_b) > 0$ both independent of k such that for any $N > N_0$ and $E \in \mathbb{R}$,

$$\begin{aligned} & |(z-c)(z-d)|^{1/2} |\tilde{m}_{h,N}(z) - \tilde{m}(z)| \leq \\ & \begin{cases} \frac{N^{\epsilon_0}}{N\eta^2} \ln N \cdot C_{ind} |(z-c)(z-d)|^{1/2} & \text{if } \eta \in (0, N^{-1/2}); \\ \frac{N^{\epsilon_0}}{N\eta^2} \ln N \cdot C_{ind} & \text{if } \eta \in [N^{-1/2}, N^{-a_k}]; \\ \frac{N^{\epsilon_0}}{N^{2a_k+\epsilon_0}} \ln N \cdot C_{ind} & \text{if } \eta \in [N^{-a_k}, 10\kappa]. \end{cases} \end{aligned}$$

The following theorem is an analogue of Lemma 6.6 of [8]. We use the same method as the proof of Lemma 6.6 of [8] to prove it.

Theorem 9.6. $\mathcal{P}(k)$ is true for all $k \in \{1, 2, \dots\}$.

Because of (9.27), there exists $N_0 > 0$ depending on ϵ_0 , V_p , κ and C_b but independent of the choice of $\{c_N\}$ such that if $N > N_0$ and $0 < d(\xi) < 9\kappa$, then

$$\begin{aligned} & \left| (\tilde{m}_{h,N}(\xi) - \tilde{m}(\xi))^2 + (2\tilde{m}(\xi) - V'_p(\xi))(\tilde{m}_{h,N}(\xi) - \tilde{m}(\xi)) + A(\xi) + B(\xi) + D(\xi) \right| \\ & \leq \frac{2}{N^2\beta} \cdot \frac{4}{\text{dist}(\xi, [a, b])} \cdot \exp\left(-\frac{N\beta}{4}\right) \leq \frac{2}{N^2\beta} \cdot \frac{4}{|\text{Im}\xi|} \cdot \exp\left(-\frac{N\beta}{4}\right) \end{aligned} \quad (9.28)$$

where

$$\begin{aligned} A(\xi) &= \frac{1}{N^2} \text{Var}_{\tilde{\mu}(h)}\left(\sum \frac{1}{\xi - \lambda_i}\right) - \frac{1}{N} \left(\frac{2}{\beta} - 1\right) \tilde{m}'_{h,N}(\xi) + (1 - c_N) V'_p(\xi) \tilde{m}_{h,N}(\xi); \\ B(\xi) &= \int_a^b \frac{V'_p(\xi) - V'_p(t)}{\xi - t} (\tilde{\rho}_1^{(h,N)}(t) - \tilde{\rho}(t)) dt + (c_N - 1) \int_a^b \frac{V'_p(\xi) - W_p^{N'}(t)}{\xi - t} \tilde{\rho}_1^{(h,N)}(t) dt, \\ D(\xi) &= \int_a^b \frac{V'_p(t) - W_p^{N'}(t)}{\xi - t} \tilde{\rho}_1^{(h,N)}(t) dt = -\frac{2}{N c_N} \int_a^b \frac{h'(t)}{\xi - t} \tilde{\rho}_1^{(h,N)}(t) dt. \end{aligned}$$

By the Hypothesis 5.1 and Lemma 8.1, $2\tilde{m}(\xi) - V'_p(\xi) \neq 0$ if $0 < \text{dist}(\xi, [c, d]) < 6\kappa$. For ξ in $\{\xi | \text{dist}(\xi, [c, d]) \in (0, 6\kappa)\}$ define

$$\bar{r}(\xi) = \frac{\sqrt{(c - \xi)(d - \xi)}}{2\tilde{m}(\xi) - V'_p(\xi)} = -\frac{1}{2\bar{r}(\xi)}$$

where $\sqrt{(c - \xi)(d - \xi)} \sim \xi$ as $|\xi| \rightarrow +\infty$.

Lemma 9.7. • $\bar{r}(\xi)$ can be continuously extended to a neighborhood of $\{z \in \mathbb{C} | \text{dist}(z, [c, d]) \leq 10\kappa\}$.

• $0 < \inf_{\text{dist}(\xi, [c, d]) < 6\kappa} |\bar{r}(\xi)| \leq \sup_{\text{dist}(\xi, [c, d]) < 6\kappa} |\bar{r}(\xi)| < +\infty.$

Remark. Lemma 9.7 can be directly induced from Condition 1 of Hypothesis 5.1.

Set $z = E + i\eta$ with $E \in \mathbb{R}$ and $\eta \in (N^{-a_k}, 10\kappa)$. Suppose $\varsigma \in (\frac{N^{-a_k}}{4}, \frac{\eta}{2})$ and $\Omega_\varsigma = \{\xi | d(\xi) \leq \varsigma\} \subset \{\xi | \text{dist}(\xi, [c, d]) < 6\kappa\}$. (Recall $d(\xi) = \text{dist}(\xi, [a, b])$ and $[a, b] = [c - \frac{\kappa}{2}, d + \frac{\kappa}{2}]$.) Suppose $T = \{\xi | |\text{Im}\xi| > N^{-1000}\}$.

From (9.28), there exists $N_0 > 0$ depending on ϵ_0 , V_p , κ and C_b such that if $N > N_0$, then

$$\begin{aligned} & \int_{T \cap \partial\Omega_\varsigma} \frac{-(\tilde{m}_{h,N}(\xi) - \tilde{m}(\xi))^2 - A(\xi)}{z - \xi} \bar{r}(\xi) d\xi - \int_{T \cap \partial\Omega_\varsigma} \frac{B(\xi) + D(\xi)}{z - \xi} \bar{r}(\xi) d\xi \\ &= \int_{T \cap \partial\Omega_\varsigma} \frac{(\tilde{m}_{h,N}(\xi) - \tilde{m}(\xi))}{z - \xi} \sqrt{(c - \xi)(d - \xi)} d\xi + \Phi_1 \end{aligned} \quad (9.29)$$

with $|\Phi_1| \leq \frac{256}{\beta} (d - c + 4) N^{1000} \exp(-\frac{N\beta}{4}) \sup_{d(\xi) \leq 1} |\bar{r}(\xi)|$. Here the direction of $T \cap \partial\Omega_\varsigma$ is induced from the counterclockwise direction of $\partial\Omega_\varsigma$.

By Lemma 9.4, there are $D_1 = D_1(V_p, \kappa, \epsilon_0, C_b) > 0$ and $N_0 = N_0(V_p, \kappa, \epsilon_0, C_b) > 0$ such that if $N > N_0$ and $\xi \in \partial\Omega_\varsigma \setminus A$, then $|\tilde{m}_{h,N}(\xi) - \tilde{m}(\xi)| \leq 4D_1N$ and $|\frac{1}{N^2} \text{Var}_{\tilde{\mu}(h)}(\sum \frac{1}{z-\lambda_k})| \leq 16D_1N^2$ and thus

$$|A(\xi)| \leq 16D_1N^2 + 16N|\frac{2}{\beta} - 1| + 4N \sup_{d(\xi) \leq 9\kappa} |V_p'(\xi)|, \quad |B(\xi)| \leq 24N \sup_{d(\xi) \leq 9\kappa} |V_p'(\xi)|, \quad |D(\xi)| \leq \frac{16\|h'\|_\infty}{\sqrt{N}}.$$

This together with (9.29) and the fact that $|\partial\Omega_\varsigma \setminus W| \leq 8N^{-1000}$ give that if $N > N_0(V_p, \kappa, \epsilon_0, C_b)$, then

$$\begin{aligned} & \int_{\partial\Omega_\varsigma} \frac{-(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 - A(\xi)}{z - \xi} \bar{r}(\xi) d\xi - \int_{\partial\Omega_\varsigma} \frac{B(\xi) + D(\xi)}{z - \xi} \bar{r}(\xi) d\xi \\ &= \int_{\partial\Omega_\varsigma} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))}{z - \xi} \sqrt{(c - \xi)(d - \xi)} d\xi + \Phi_2 \end{aligned}$$

with $|\Phi_2| \leq D_2N^{-997}$ and D_2 depends on V_p, κ, ϵ_0 and C_b . Here the direction of $\partial\Omega_\varsigma$ in each integral is counterclockwise.

Notice that $|\tilde{m}_{h,N}(\xi) - \tilde{m}(\xi)| = |\int \frac{1}{\xi-t} (\tilde{\rho}_1^{(h,N)}(t) - \tilde{\rho}(t)) dt| = |\int \frac{t}{\xi(\xi-t)} (\tilde{\rho}_1^{(h,N)}(t) - \tilde{\rho}(t)) dt| = O(|\xi|^{-2})$ as $|\xi| \rightarrow \infty$. This together with the analyticity of $\sqrt{(c - \xi)(d - \xi)}$ on $\mathbb{C} \setminus [c, d]$ and Cauchy's integral formula give:

$$\int_{\partial\Omega_\varsigma} \frac{(\tilde{m}_{h,N}(\xi) - \tilde{m}(\xi))}{z - \xi} \sqrt{(c - \xi)(d - \xi)} d\xi = 2\pi i (\tilde{m}_{h,N}(z) - \tilde{m}(z)) \sqrt{(c - z)(d - z)}.$$

This equation is important. It tells us that although $\sqrt{(c - z)(d - z)}$ can be arbitrarily large when $|z| \rightarrow \infty$, $(\tilde{m}_{h,N}(z) - \tilde{m}(z)) \sqrt{(c - z)(d - z)}$ is still controlled by the integral along $\partial\Omega_\varsigma$ on the left hand side. This equation was first induced in the proof of Lemma 6.6 of [8].

In other words, we have the following proposition.

Proposition 9.1. *There exist $D_2 = D_2(V_p, \kappa, \epsilon_0, C_b) > 0$, $N_0 = N_0(V_p, \kappa, \epsilon_0, C_b) > 0$ such that if $N > N_0$, $z = E + i\eta$ with $E \in \mathbb{R}$, $\eta \in (N^{-a_k}, 10\kappa)$, $\varsigma \in (\frac{N^{-a_k}}{4}, \frac{\eta}{2})$ and $\Omega_\varsigma = \{\xi | d(\xi) \leq \varsigma\}$, then*

$$\begin{aligned} & \sqrt{(c - z)(d - z)} (\tilde{m}_{h,N}(z) - \tilde{m}(z)) \\ &= -\frac{1}{2\pi i} \int_{\partial\Omega_\varsigma} \frac{(\tilde{m}_{h,N}(\xi) - \tilde{m}(\xi))^2 + A(\xi)}{z - \xi} \bar{r}(\xi) d\xi - \frac{1}{2\pi i} \int_{\partial\Omega_\varsigma} \frac{B(\xi) + D(\xi)}{z - \xi} \bar{r}(\xi) d\xi - \frac{1}{2\pi i} \cdot \Phi_2. \end{aligned} \quad (9.30)$$

where

$$\begin{aligned} A(\xi) &= \frac{1}{N^2} \text{Var}_{\tilde{\mu}(h)}(\sum \frac{1}{\xi - \lambda_i}) - \frac{1}{N} (\frac{2}{\beta} - 1) \tilde{m}'_{h,N}(\xi) + (1 - c_N) V_p'(\xi) \tilde{m}_{h,N}(\xi); \\ B(\xi) &= \int_a^b \frac{V_p'(\xi) - V_p'(t)}{\xi - t} (\tilde{\rho}_1^{(h,N)}(t) - \tilde{\rho}(t)) dt + (c_N - 1) \int_a^b \frac{V_p'(\xi) - V_p'(t)}{\xi - t} \tilde{\rho}_1^{(h,N)}(t) dt; \\ D(\xi) &= -\frac{2}{Nc_N} \int_a^b \frac{h'(t)}{\xi - t} \tilde{\rho}_1^{(h,N)}(t) dt; \\ |\Phi_2| &\leq D_2N^{-997}. \end{aligned}$$

In the following two subsections we estimate the first two terms on the right hand side of (9.30).

9.2.1 Estimation of the first term on the right hand side of (9.30)

Using Lemma 9.4 we can easily verify that there are $D_3 > 0$, $N_0 > 0$ depending on V_p , κ , ϵ_0 and C_b such that if $N > N_0$ and $\xi \in \partial\Omega_\varsigma$, then $|(\tilde{m}_{h,N}(\xi) - \tilde{m}(\xi))^2 + A(\xi)| \leq D_3(\frac{1}{d(\xi)^2} \frac{\ln N}{N} + \frac{1}{d(\xi)} N^{-1+\epsilon_0})$ and therefore

$$\left| -\frac{1}{2\pi i} \int_{\partial\Omega_\varsigma} \frac{(\tilde{m}_{h,N}(\xi) - \tilde{m}(\xi))^2 + A(\xi)}{z - \xi} \bar{r}(\xi) d\xi \right| \leq \frac{D_3}{2\pi} \left(\sup_{d(\xi) \leq 1} |\bar{r}(\xi)| \right) \left(\frac{1}{\varsigma^2} \frac{\ln N}{N} + \frac{1}{\varsigma} \frac{N^{\epsilon_0}}{N} \right) \int_{\partial\Omega_\varsigma} \frac{1}{|z - \xi|} d\xi.$$

It is not hard to check that there exist $N_0 = N_0(V_p, \kappa, \epsilon_0) > 0$ such that if $N > N_0$, $z = E + i\eta$ with $E \in \mathbb{R}$, $\eta \in (N^{-a_k}, 10\kappa)$ and $\varsigma \in (\frac{N^{-a_k}}{4}, \frac{\eta}{2})$, then

$$\left| \int_{\partial\Omega_\varsigma} \frac{1}{|z - \xi|} d\xi \right| \leq 12 \ln N. \quad (9.31)$$

Thus if $N > N_0(V_p, \kappa, \epsilon_0, C_b)$, $E \in \mathbb{R}$, $\eta \in (N^{-a_k}, 10\kappa)$, $\varsigma \in (\frac{N^{-a_k}}{4}, \frac{\eta}{2})$, then the first term on the right hand side of (9.30) is no more than

$$\frac{6D_3}{\pi} \left(\sup_{d(\xi) \leq 1} |\bar{r}(\xi)| \right) \left(\frac{1}{\varsigma^2} \frac{\ln N}{N} + \frac{1}{\varsigma} \frac{N^{\epsilon_0}}{N} \right) \ln N$$

and D_3 depends on V_p , κ , ϵ_0 and C_b .

9.2.2 Estimation of the second term on the right hand side of (9.30)

Set $b_N(\xi) = \int_a^b \frac{V_p'(\xi) - V_p'(t)}{\xi - t} (\tilde{\rho}_1^{(h,N)}(t) - \tilde{\rho}(t)) dt$. Obviously $\xi \mapsto b_N(\xi)$ is analytic on $\mathbb{C} \setminus \mathbb{R}$. For $z = E + i\eta$ with $E \in \mathbb{R}$, $\eta \in (N^{-a_k}, 10\kappa)$ and $\varsigma \in (\frac{N^{-a_k}}{4}, \frac{\eta}{2})$, we have

$$\left| \int_{\partial\Omega_\varsigma} \frac{b_N(\xi)}{z - \xi} \bar{r}(\xi) d\xi \right| = \lim_{s \rightarrow 0^+} \left| \int_{\partial\Omega_\varsigma \cap \{|\operatorname{Im}\xi| > s\}} \frac{b_N(\xi)}{z - \xi} \bar{r}(\xi) d\xi \right| = 2 \left| \iint_{\Omega_\varsigma \setminus \mathbb{R}} \partial_{\bar{\xi}} \left(\frac{b_N(\xi)}{z - \xi} \bar{r}(\xi) \right) dx dy \right|.$$

where $\partial_{\bar{\xi}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) = \frac{1}{2}(\frac{\partial}{\partial E} + i\frac{\partial}{\partial \eta})$. Here we used Green's formula. (To use Green's formula we need to check that the real part and imaginary part of $\frac{b_N(\xi)}{z - \xi} \bar{r}(\xi)$ are both C^1 on a neighborhood of $\Omega_\varsigma \cap \{|\operatorname{Im}\xi| > s\}$ for all $s > 0$. This is ensured by: i) $b_N(\xi)$ is analytic on $\mathbb{C} \setminus \mathbb{R}$, ii) $\bar{r}(\xi)$ is analytic on Ω_ς .)

Since both $b_N(\xi)$ and $\bar{r}(\xi)$ are analytic on $\Omega_\varsigma \setminus \mathbb{R}$ and are independent of $\bar{\xi}$, we have $\partial_{\bar{\xi}} \left(\frac{b_N(\xi)}{z - \xi} \bar{r}(\xi) \right) = 0$ on $\Omega_\varsigma \setminus \mathbb{R}$.

So under the assumption of Proposition 9.1,

$$\begin{aligned} \left| -\frac{1}{2\pi i} \int_{\partial\Omega_\varsigma} \frac{B(\xi)}{z - \xi} \bar{r}(\xi) d\xi \right| &= \left| -\frac{1}{2\pi i} \int_{\partial\Omega_\varsigma} \frac{b_N(\xi)}{z - \xi} \bar{r}(\xi) d\xi - \frac{1}{2\pi i} \int_{\partial\Omega_\varsigma} \frac{\bar{r}(\xi)}{z - \xi} (c_N - 1) \int_a^b \frac{V_p'(\xi) - V_p'(t)}{\xi - t} \tilde{\rho}_1^{(h,N)}(t) dt d\xi \right| \\ &= \frac{|c_N - 1|}{2\pi} \left| \int_{\partial\Omega_\varsigma} \frac{\bar{r}(\xi)}{z - \xi} \int_a^b \frac{V_p'(\xi) - V_p'(t)}{\xi - t} \tilde{\rho}_1^{(h,N)}(t) dt d\xi \right| \\ &\leq \frac{2}{\pi} (b - a + 4) N^{-1+\epsilon_0} \frac{1}{\varsigma^2} \left(\sup_{d(\xi) \leq 6\kappa} |V_p'(\xi)| \right) \left(\sup_{d(\xi) \leq 1} |\bar{r}(\xi)| \right). \end{aligned}$$

$$\left| -\frac{1}{2\pi i} \int_{\partial\Omega_\varsigma} \frac{D(\xi)}{z-\xi} \bar{r}(\xi) d\xi \right| \leq \frac{1}{2\pi} \left| \int_{\partial\Omega_\varsigma} \frac{1}{z-\xi} \frac{4}{N} \frac{\|h'\|_\infty}{\varsigma} \bar{r}(\xi) d\xi \right| \leq \frac{24}{\pi} \|h'\|_\infty \left(\sup_{d(\xi) \leq 1} |\bar{r}(\xi)| \right) \frac{\ln N}{N} \frac{1}{\varsigma}.$$

The last step comes from (9.31).

9.2.3 Conclusion

Proposition 9.1 and the arguments in Subsection 9.2.1 and 9.2.2 implies the following lemma.

Lemma 9.8. *There exist $D_4 = D_4(V_p, \kappa, \epsilon_0, C_b) > 0$, $N_0 = N_0(V_p, \kappa, \epsilon_0, C_b) > 0$ such that if $N > N_0$, $z = E + i\eta$ with $E \in \mathbb{R}$, $\eta \in (N^{-a_k}, 10\kappa)$, $\varsigma \in (\frac{N^{-a_k}}{4}, \frac{\eta}{2})$, then*

$$|\sqrt{(c-z)(d-z)}(\tilde{m}_{h,N}(z) - \tilde{m}(z))| \leq D_4 \left(\frac{N^{\epsilon_0}}{N} \left(\frac{1}{\varsigma^2} + \frac{\ln N}{\varsigma} \right) + N^{-997} \right).$$

Proof of Theorem 9.6. From Lemma 9.5 we know $\mathcal{P}(1)$ is correct for $\eta \in (0, N^{-1/2})$. Setting $\varsigma = \frac{\eta}{2}$ in Lemma 9.8 we know $\mathcal{P}(1)$ is correct for $\eta \in [N^{-1/2}, 10\kappa)$. $\mathcal{P}(2)$ can be proved in a similar way. Suppose $k \geq 2$ and $\mathcal{P}(k)$ is correct. Now we consider $\mathcal{P}(k+1)$. From $\mathcal{P}(k)$ we see $\mathcal{P}(k+1)$ is true for $\eta \in (0, N^{-a_k})$. So we only consider $\eta \in (N^{-a_k}, 10\kappa)$. Recall that $E \in \mathbb{R}$ and $z = E + i\eta$.

If $\eta \in (N^{-a_k}, N^{-a_k/2})$, set $\varsigma = \eta/2$. From Lemma 9.8, if $N > N_0(V_p, \kappa, \epsilon_0, C_b)$, then

$$|\sqrt{(c-z)(d-z)}(\tilde{m}_{h,N}(z) - \tilde{m}(z))| \leq 7D_4 \frac{N^{\epsilon_0}}{N\eta^2} \ln N.$$

If $\eta \in [N^{-a_k/2}, 10\kappa)$, set $\varsigma = \frac{1}{2}N^{-a_k/2}$. From Lemma 9.8, if $N > N_0(V_p, \kappa, \epsilon_0, C_b)$, then

$$|\sqrt{(c-z)(d-z)}(\tilde{m}_{h,N}(z) - \tilde{m}(z))| \leq 7D_4 \frac{N^{a_k+\epsilon_0}}{N} \ln N.$$

Since $a_{k+1} = \frac{1}{2}a_k$ and both D_4 and N_0 are independent of k , the proof is complete. \square

9.3 Proof of Theorem 9.1

We need the following lemma about Helffer-Sjostrand calculus.

Lemma 9.9. *Suppose $\chi(x) : \mathbb{R} \rightarrow [-1, 1]$ is a smooth even function with $\chi(x) = 1$ on $[-\frac{1}{2}D_\chi, \frac{1}{2}D_\chi]$ and $\chi(x) = 0$ on $[-D_\chi, D_\chi]$ where $D_\chi > 0$. Suppose $f(x)$ is a C^2 function with compact support. Suppose $\bar{\rho}(t)dt$ is a signed measure with Stieltjes transform $\bar{m}(x+iy) = \int \frac{1}{x+iy-t} \bar{\rho}(t)dt$. Suppose $\int |\bar{\rho}(t)|dt < +\infty$. We have*

$$\begin{aligned} \left| \int f(t) \bar{\rho}(t) dt \right| &\leq \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} y f''(x) \chi(y) \operatorname{Im} \bar{m}(x+iy) dx dy \right| \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{R}^2} (|f(x)| + |y| |f'(x)|) |\chi'(y)| |\bar{m}(x+iy)| dx dy \end{aligned}$$

A brief proof of Lemma 9.9 was given as [B.13] in [13]. For the convenience of readers we write a proof with more details here.

Proof of Lemma 9.9. Suppose the support of f is contained in $[-D_f, D_f]$. Set $z = x + iy$ and $\tilde{f}(z) = \tilde{f}(x + iy) = (f(x) + iyf'(x))\chi(y)$. Let $M = \{x + iy | x \in [-D_f, D_f], y \in [-D_\chi, D_\chi]\}$. So $\tilde{f}(z) \neq 0$ only when $z \in M$.

Suppose $\lambda \in \mathbb{R}$ and $B(\lambda, c) = \{z \in \mathbb{C} | \text{dist}(z, \lambda) < c\}$. Set $\partial B(\lambda, c)$ be the boundary of $B(\lambda, c)$ with counterclockwise direction. Since

$$\int_{\partial B(\lambda, c)} \frac{\tilde{f}(z)}{z - \lambda} dz = \int_0^{2\pi} \frac{\tilde{f}(\lambda + re^{i\theta})}{re^{i\theta}} d(\lambda + re^{i\theta}) = i \int_0^{2\pi} \tilde{f}(\lambda + re^{i\theta}) d\theta,$$

we have

$$\lim_{c \rightarrow 0} \int_{\partial B(\lambda, c)} \frac{\tilde{f}(z)}{z - \lambda} dz = 2\pi i f(\lambda).$$

Suppose $L > 0$ is large enough such that M is in the interior of $B(\lambda, L)$. Suppose $\partial B(\lambda, L)$ is the boundary of $B(\lambda, L)$ with counterclockwise direction.

$$\text{Set } \partial_{\bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Lemma 9.10. *We have*

$$\int_{\mathbb{R}^2} \left| \frac{\partial_{\bar{z}} \tilde{f}(z)}{z - \lambda} \right| dx dy < +\infty.$$

Proof of Lemma 9.10. It suffices to prove that

$$\int_M \frac{|yf''(x)\chi(y)|}{|z - \lambda|} dx dy < +\infty, \quad \int_M \frac{|f(x)\chi'(y)|}{|z - \lambda|} dx dy < +\infty, \quad \int_M \frac{|yf'(x)\chi'(y)|}{|z - \lambda|} dx dy < +\infty.$$

If $\chi'(y) \neq 0$, then $\frac{1}{2}D_\chi < |y| < D_\chi$ and $|z - \lambda| \geq |y| > \frac{1}{2}D_\chi$. So the second and third inequalities are true. To show the first one we only need to prove

$$\int_M \frac{|x - \lambda|}{(x - \lambda)^2 + y^2} dx dy < +\infty \quad \text{and} \quad \int_M \frac{|y|}{(x - \lambda)^2 + y^2} dx dy < +\infty$$

while both of them can be induced from

$$\int_{-A}^A \int_{-A}^A \frac{|x|}{x^2 + y^2} dx dy = \int_{-A}^A 2 \arctan \frac{A}{|x|} dx \leq 2\pi A, \quad \forall A > 0.$$

□

Let's continue to prove Lemma 9.9.

Thus by Green's formula,

$$\int_{\mathbb{R}^2} \frac{\partial_{\bar{z}} \tilde{f}(z)}{z - \lambda} dx dy = \lim_{c \rightarrow 0} \int_{B(\lambda, L) \setminus B(\lambda, c)} \frac{\partial_{\bar{z}} \tilde{f}(z)}{z - \lambda} dx dy = -\frac{1}{2i} \lim_{c \rightarrow 0} \int_{\partial B(\lambda, c)} \frac{\tilde{f}(z)}{z - \lambda} dz = -\pi f(\lambda).$$

Therefore

$$\begin{aligned}
\left| \int_{\mathbb{R}} f(t) \bar{\rho}(t) dt \right| &= \frac{1}{\pi} \left| \operatorname{Re} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{\partial_{\bar{z}} \tilde{f}(z)}{z-t} dx dy \bar{\rho}(t) dt \right| \\
&= \frac{1}{\pi} \left| \operatorname{Re} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \frac{\partial_{\bar{z}} \tilde{f}(z)}{z-t} \bar{\rho}(t) dt dx dy \right| \quad (\text{by Fubini's Theorem}) \\
&= \frac{1}{\pi} \left| \operatorname{Re} \int_{\mathbb{R}^2} \partial_{\bar{z}} \tilde{f}(z) \bar{m}(x+iy) dx dy \right| \\
&= \frac{1}{2\pi} \left| \operatorname{Re} \int_{\mathbb{R}^2} \left(iy f''(x) \chi(y) + if(x) \chi'(y) - y f'(x) \chi'(y) \right) \bar{m}(x+iy) dx dy \right| \\
&\leq \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} y f''(x) \chi(y) \operatorname{Im} \bar{m}(x+iy) dx dy \right| + \frac{1}{2\pi} \int_{\mathbb{R}^2} (|f(x) \chi'(y)| + |y f'(x) \chi'(y)|) |\bar{m}(x+iy)| dx dy
\end{aligned}$$

□

Now we use a similar method as the proof of Lemma 6.5 of [8] to prove Theorem 9.1. Recall that $\tilde{\mu}$ is a probability measure on $[a, b]^N$ with density

$$\frac{1}{Z_{\tilde{\mu}}} \exp\left(-\frac{\beta N}{2} \sum_{i=1}^N c_N V_p(x_i)\right) \prod_{i < j} |x_i - x_j|^\beta \prod_{i=1}^N \mathbf{1}_{[a,b]}(x_i)$$

and equilibrium measure $\tilde{\rho}(t)dt$. Recall that $h : \mathbb{R} \rightarrow \mathbb{R}$ is C^2 and $\|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty < C_b$. Suppose $\tau > 0$.

Notice that in Theorem 9.1

$$\sum_{i=1}^N h(x_i) - N \int h(x) \tilde{\rho}(x) dx = N \int h(x) \tilde{\rho}_1^{(h,N)}(x) dx - N \int h(x) \tilde{\rho}(x) dx$$

and the integrals are both over $[a, b]$. So without loss of generality we suppose h is compactly supported.

Suppose $k > 1$. Recall that $a_k = (\frac{1}{2})^k$. Suppose $\chi(x) : \mathbb{R} \rightarrow [-1, 1]$ is a smooth even function with $\chi(x) = 1$ on $[-\frac{1}{2}N^{-a_k}, \frac{1}{2}N^{-a_k}]$, $\chi(x) = 0$ on $[-N^{-a_k}, N^{-a_k}]^c$ and $|\chi'| \leq 100N^{a_k}$.

Suppose $s \in (-\beta, \beta)$ and $\bar{\rho}(t) = \tilde{\rho}_1^{(-\frac{s}{\beta}h, N)}(t) - \tilde{\rho}(t)$.

By Lemma 9.9,

$$\begin{aligned}
\left| \int h(x) (\tilde{\rho}_1^{(-\frac{s}{\beta}h, N)}(x) - \tilde{\rho}(x)) dx \right| &\leq \frac{2C_b}{\pi} \iint_{\substack{x \in [a,b] \\ 0 < \eta < N^{-a_k}}} (\eta \chi(\eta) + |\chi'(\eta)|) |\tilde{m}_{-\frac{s}{\beta}h, N}(x+i\eta) - \tilde{m}(x+i\eta)| dx d\eta \\
&= \frac{2C_b}{\pi} (I + II + III)
\end{aligned}$$

where

$$I = \iint_{\substack{a < x < b \\ \frac{1}{2}N^{-a_k} < \eta < N^{-a_k}}} |\chi'(\eta)| |\tilde{m}_{-\frac{s}{\beta}h, N}(x+i\eta) - \tilde{m}(x+i\eta)| dx d\eta$$

$$\begin{aligned}
II &= \iint_{\substack{a < x < b \\ 0 < \eta < N^{-\frac{1}{2}}}} \eta \chi(\eta) |\tilde{m}_{-\frac{s}{\beta}h, N}(x + i\eta) - \tilde{m}(x + i\eta)| dx d\eta \\
III &= \iint_{\substack{a < x < b \\ N^{-\frac{1}{2}} < \eta < N^{-a_k}}} \eta \chi(\eta) |\tilde{m}_{-\frac{s}{\beta}h, N}(x + i\eta) - \tilde{m}(x + i\eta)| dx d\eta
\end{aligned}$$

9.3.1 Estimation of I

Notice $|(x + i\eta - c)(x + i\eta - d)| \geq \frac{d-c}{2}\eta$. From Theorem 9.6, there are $C_8 > 0$, $N_0 > 0$ depending on V , β , κ and C_b such that for $N > N_0$

$$\begin{aligned}
I &= \iint_{\substack{a < x < b \\ \frac{1}{2}N^{-a_k} < \eta < N^{-a_k}}} |\chi'(\eta)| |\tilde{m}_{-\frac{s}{\beta}h, N}(x + i\eta) - \tilde{m}(x + i\eta)| dx d\eta \\
&\leq (b-a) 100 N^{a_k} \int_{\frac{1}{2}N^{-a_k}}^{N^{-a_k}} \frac{N^{\epsilon_0}}{N\eta^2} \ln N \cdot C_{ind} \frac{1}{\sqrt{\frac{d-c}{2}\eta}} d\eta \\
&= (b-a) 100 N^{-1+a_k+\epsilon_0} \ln N \cdot C_{ind} \sqrt{\frac{2}{d-c}} \left(-\frac{2}{3}\right) (1 - 2\sqrt{2}) N^{\frac{3}{2}a_k} \\
&\leq C_8 N^{-1+\frac{s}{2}a_k+\epsilon_0} \ln N.
\end{aligned}$$

9.3.2 Estimation of II

By Lemma 9.4, there are $C_9 > 0$, $N_0 > 0$ depending on V , β , κ and C_b such that for $N > N_0$,

$$\begin{aligned}
II &= \iint_{\substack{a < x < b \\ 0 < \eta < N^{-\frac{1}{2}}}} \eta \chi(\eta) |\tilde{m}_{-\frac{s}{\beta}h, N}(x + i\eta) - \tilde{m}(x + i\eta)| dx d\eta \leq \iint_{\substack{a < x < b \\ 0 < \eta < N^{-\frac{1}{2}}}} \eta \chi(\eta) \frac{C_9}{\eta} \sqrt{\frac{\ln N}{N}} dx d\eta \\
&\leq C_9 (b-a) \frac{\sqrt{\ln N}}{N}.
\end{aligned}$$

9.3.3 Estimation of III

$$\begin{aligned}
III &= \iint_{\substack{a < x < b \\ N^{-\frac{1}{2}} < \eta < N^{-a_k}}} \eta \chi(\eta) |\tilde{m}_{-\frac{s}{\beta}h, N}(x + i\eta) - \tilde{m}(x + i\eta)| dx d\eta \\
&= \iint_{\substack{a < x < \frac{c+d}{2} \\ N^{-\frac{1}{2}} < \eta < N^{-a_k}}} \eta \chi(\eta) |\tilde{m}_{-\frac{s}{\beta}h, N}(x + i\eta) - \tilde{m}(x + i\eta)| dx d\eta + \iint_{\substack{\frac{c+d}{2} < x < b \\ N^{-\frac{1}{2}} < \eta < N^{-a_k}}} \eta \chi(\eta) |\tilde{m}_{-\frac{s}{\beta}h, N}(x + i\eta) - \tilde{m}(x + i\eta)| dx d\eta \\
&:= R_1 + R_2
\end{aligned}$$

Notice $|(x + i\eta - c)(x + i\eta - d)| \geq \frac{b-a}{2}|x + i\eta - c|$ for $a < x < \frac{c+d}{2}$. From Theorem 9.6, there are $C_{ind}(V_p, \epsilon_0, \kappa, C_b) > 0$ and $N_0(V_p, \epsilon_0, \kappa, C_b, k) > 0$ such that for $N > N_0$

$$\begin{aligned}
R_1 &= \iint_{\substack{a < x < \frac{c+d}{2} \\ N^{-\frac{1}{2}} < \eta < N^{-a_k}}} \eta \chi(\eta) |\tilde{m}_{-\frac{s}{\beta}h, N}(x + i\eta) - \tilde{m}(x + i\eta)| dx d\eta \\
&\leq \iint_{\substack{a < x < \frac{c+d}{2} \\ N^{-\frac{1}{2}} < \eta < N^{-a_k}}} \eta \frac{1}{\sqrt{|x + i\eta - c|^{\frac{b-a}{2}}}} \frac{N^{\epsilon_0}}{N\eta^2} \ln N \cdot C_{ind} dx d\eta \\
&\leq \iint_{\substack{a < x < \frac{c+d}{2} \\ N^{-\frac{1}{2}} < \eta < N^{-a_k}}} \eta \frac{1}{\sqrt{\max(|x - c|, \eta)^{\frac{b-a}{2}}}} \frac{N^{\epsilon_0}}{N\eta^2} \ln N \cdot C_{ind} dx d\eta \\
&= \sqrt{\frac{2}{b-a}} \frac{N^{\epsilon_0}}{N} \ln N \cdot C_{ind} \left(\iint_{K_1} \frac{1}{\eta \sqrt{|x - c|}} dx d\eta + \iint_{K_2} \eta^{-\frac{3}{2}} dx d\eta \right)
\end{aligned}$$

where

$$\begin{aligned}
K_2 &= \{x + i\eta | a < x < \frac{c+d}{2}, N^{-\frac{1}{2}} < \eta < N^{-a_k}, |x - c| < \eta\} \\
&= \{x + i\eta | c - \eta < x < c + \eta, N^{-\frac{1}{2}} < \eta < N^{-a_k}\}
\end{aligned}$$

$$\begin{aligned}
K_1 &= \{x + i\eta | a < x < \frac{c+d}{2}, N^{-\frac{1}{2}} < \eta < N^{-a_k}, |x - c| > \eta\} \\
&= \{x + i\eta | a < x < c - \eta, N^{-\frac{1}{2}} < \eta < N^{-a_k}\} \cup \{x + i\eta | c + \eta < x < \frac{c+d}{2}, N^{-\frac{1}{2}} < \eta < N^{-a_k}\} \\
&:= K_3 \cup K_4.
\end{aligned}$$

(Here we used the fact that $\eta < \min(c - a, \frac{d-c}{2})$ when $N > N_0(k)$.)

Consider the integral on K_2 :

$$\iint_{K_2} \eta^{-\frac{3}{2}} dx d\eta = \int_{N^{-\frac{1}{2}}}^{N^{-a_k}} 2\eta^{-\frac{1}{2}} d\eta = 4N^{-\frac{a_k}{2}} - 4N^{-\frac{1}{4}}.$$

Consider the integral on K_3 :

$$\iint_{K_3} \frac{1}{\eta \sqrt{|x - c|}} dx d\eta = \int_{N^{-\frac{1}{2}}}^{N^{-a_k}} 2\sqrt{c - a} \eta^{-1} - 2\eta^{-\frac{1}{2}} d\eta = 2\sqrt{c - a} \left(\frac{1}{2} - a_k\right) \ln N - 4N^{-\frac{a_k}{2}} + 4N^{-\frac{1}{4}}.$$

Consider the integral on K_4 :

$$\iint_{K_4} \frac{1}{\eta \sqrt{|x - c|}} dx d\eta = \int_{N^{-\frac{1}{2}}}^{N^{-a_k}} 2\sqrt{\frac{d-c}{2}} \eta^{-1} - 2\eta^{-\frac{1}{2}} d\eta = 2\sqrt{\frac{d-c}{2}} \left(\frac{1}{2} - a_k\right) \ln N - 4N^{-\frac{a_k}{2}} + 4N^{-\frac{1}{4}}.$$

Thus the sum of the integrals over K_2 , K_3 and K_4 is $(1 - 2a_k)(\sqrt{c - a} + \sqrt{\frac{d - c}{2}}) \ln N - 4N^{-\frac{a_k}{2}} + 4N^{-\frac{1}{4}}$. So there are $C_{10}(V_p, \kappa, \epsilon_0, C_b) > 0$ and $N_0(V_p, \kappa, \epsilon_0, C_b, k) > 0$ such that for $N > N_0$

$$\begin{aligned} R_1 &\leq \sqrt{\frac{2}{b - a}} \frac{N^{\epsilon_0}}{N} \ln N \cdot C_{ind} \left((1 - 2a_k)(\sqrt{c - a} + \sqrt{\frac{d - c}{2}}) \ln N - 4N^{-\frac{a_k}{2}} + 4N^{-\frac{1}{4}} \right) \\ &\leq C_{10} \frac{N^{\epsilon_0}}{N} (\ln N)^2 \end{aligned}$$

By the same argument, R_2 has the same bound. So there are $C_{10}(V_p, \kappa, \epsilon_0, C_b) > 0$ and $N_0(V_p, \kappa, \epsilon_0, C_b, k) > 0$ such that for $N > N_0$, we have $R_2 \leq C_{10} \frac{N^{\epsilon_0}}{N} (\ln N)^2$

$$III \leq 2C_{10} \frac{N^{\epsilon_0}}{N} (\ln N)^2.$$

9.3.4 Conclusion

From the above sections, there are $C_{11}(V_p, \kappa, \epsilon_0, C_b) > 0$ and $N_0(V_p, \kappa, \epsilon_0, C_b, k) > 0$ such that for $N > N_0$

$$\left| \int h(x) (\rho_1^{(-\frac{s}{\beta} h, N)}(x) - \rho(x)) dx \right| \leq C_{11} N^{-1 + \epsilon_0 + \frac{5}{2} a_k} \ln N. \quad (9.32)$$

By direct computation,

$$\begin{aligned} \frac{d}{ds} \ln \mathbb{E}^{\tilde{\mu}} \left(e^{s(\sum h(\lambda_k) - N \int h(u) \tilde{\rho}(u) du)} \right) &= E^{\tilde{\mu}(-\frac{s}{\beta} h)} \left(\sum h(\lambda_k) - N \int h(u) \tilde{\rho}(u) du \right) \\ &= N \int h(u) (\tilde{\rho}_1^{(-\frac{s}{\beta} h, N)}(u) - \tilde{\rho}(u)) du. \end{aligned} \quad (9.33)$$

Set $f(s) = \mathbb{E}^{\tilde{\mu}} \left(e^{s(\sum h(\lambda_k) - N \int h(u) \tilde{\rho}(u) du)} \right)$. From (9.32) and (9.33), if $N > N_0(V_p, \kappa, \epsilon_0, C_b, k)$

$$\left| \frac{d}{ds} \ln f(s) \right| \leq C_{11} N^{\epsilon_0 + \frac{5}{2} a_k} \ln N.$$

Since $\ln f(0) = 0$, both $|\ln f(1)|$ and $|\ln f(-1)|$ are bounded by

$$C_{11} N^{\epsilon_0 + \frac{5}{2} a_k} \ln N$$

and when $N > N_0(V_p, \kappa, \epsilon_0, C_b, k)$

$$\begin{aligned} \mathbb{E}^{\tilde{\mu}} \left(e^{\sum h(\lambda_k) - N \int h(u) \tilde{\rho}(u) du} \right) + \mathbb{E}^{\tilde{\mu}} \left(e^{-\sum h(\lambda_k) - N \int h(u) \tilde{\rho}(u) du} \right) &= f(1) + f(-1) \\ &\leq 2 \exp [C_{11} N^{\epsilon_0 + \frac{5}{2} a_k} \ln N]. \end{aligned} \quad (9.34)$$

Suppose $0 < w_0 < \epsilon_0 + 3a_k$. If $\mathbb{P}^{\tilde{\mu}}(|\sum h(\lambda_k) - N \int h(u) \tilde{\rho}(u) du| > N^{\epsilon_0 + 3a_k}) > \exp(-N^{w_0})$, then either

$$\mathbb{P}^{\tilde{\mu}} \left(\sum h(\lambda_k) - N \int h(u) \tilde{\rho}(u) du > N^{\epsilon_0 + 3a_k} \right) > \frac{1}{2} \exp(-N^{w_0}) \quad (9.35)$$

or

$$\mathbb{P}^{\tilde{\mu}}\left(\sum h(\lambda_k) - N \int h(u) \tilde{\rho}(u) du < -N^{\epsilon_0+3a_k}\right) > \frac{1}{2} \exp(-N^{w_0}). \quad (9.36)$$

If (9.35) happens, then

$$\mathbb{E}^{\tilde{\mu}}\left(e^{\sum h(\lambda_k) - N \int h(u) \tilde{\rho}(u) du}\right) \geq \frac{1}{2} \exp(-N^{w_0}) \exp(N^{\epsilon_0+3a_k}) = \frac{1}{2} \exp(N^{\epsilon_0+3a_k} - N^{w_0})$$

which is contradictory to (9.34) when $N > N_0(V_p, \kappa, \epsilon_0, C_b, k, w_0)$.

If (9.36) happens, then

$$\mathbb{E}^{\tilde{\mu}}\left(e^{-[\sum h(\lambda_k) - N \int h(u) \tilde{\rho}(u) du]}\right) \geq \frac{1}{2} \exp(-N^{w_0}) \exp(N^{\epsilon_0+3a_k}) = \frac{1}{2} \exp(N^{\epsilon_0+3a_k} - N^{w_0})$$

which is contradictory to (9.34) when $N > N_0(V_p, \kappa, \epsilon_0, C_b, k, w_0)$.

So we have the following proposition.

Proposition 9.2. *Suppose $k \in \{2, 3, \dots\}$, $a_k = (\frac{1}{2})^k$ and $0 < w_0 < \epsilon_0 + 3a_k$. There exists $N_0(V_p, \kappa, \epsilon_0, C_b, k, w_0) > 0$ such that if $N > N_0$, then*

$$\mathbb{P}^{\tilde{\mu}}(|\sum h(\lambda_k) - N \int h(u) \tilde{\rho}(u) du| > N^{\epsilon_0+3a_k}) \leq \exp(-N^{w_0}).$$

Since k can be arbitrarily large and a_k can be arbitrarily small, we see Theorem 9.1 is a direct corollary of Proposition 9.2.

10 Analogue of Lemma 2.2 of [7]

In this section we prove a result which is an analogue of Lemma 2.2 of [7]. We use the same method as the proof of Lemma 2.2 of [7]. This result will be an useful tool for later sections.

Suppose $\tilde{\rho}_1^{(N)}(x)$ and $\tilde{\rho}_2^{(N)}(x, y)$ are the one-point and two-points correlation functions of $\tilde{\mu}$. Set $\tilde{m}_N(z) = \int \frac{1}{z-t} \tilde{\rho}_1^{(N)}(t) dt = \frac{1}{N} \mathbb{E}^{\tilde{\mu}} \sum \frac{1}{z-\lambda_i}$ and $\text{Var}_{\tilde{\mu}}(\sum \frac{1}{z-\lambda_i}) = \mathbb{E}^{\tilde{\mu}}[(\sum \frac{1}{z-\lambda_i})^2] - (\mathbb{E}^{\tilde{\mu}} \sum \frac{1}{z-\lambda_i})^2$.

From (9.25) and (9.26) with $h \equiv 0$, there exists $N_0 > 0$ depending on ϵ_0, V_p, κ , but independent of the choice of $\{c_N\}$ such that if $N > N_0$ and $z = E + i\eta \in \mathbb{C} \setminus ([a, b] \cup (-\infty, W_L] \cup [W_R, +\infty))$, then

$$\begin{aligned} & \left| (\tilde{m}_N(z) - \tilde{m}(z))^2 + (2\tilde{m}(z) - V_p'(z))(\tilde{m}_N(z) - \tilde{m}(z)) + \int_a^b \frac{V_p'(z) - V_p'(t)}{z-t} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) dt \right. \\ & \left. + (1 - c_N)V_p'(z)\tilde{m}_N(z) + (c_N - 1) \int_a^b \frac{V_p'(z) - V_p'(t)}{z-t} \tilde{\rho}_1^{(N)}(t) dt + \frac{1}{N^2} \text{Var}_{\tilde{\mu}}\left(\sum \frac{1}{z-\lambda_i}\right) - \frac{1}{N} \left(\frac{2}{\beta} - 1\right) m'_N(z) \right| \\ & \leq \frac{2}{N^2\beta} \cdot \frac{4}{\text{dist}(z, [a, b])} \cdot \exp\left(-\frac{N\beta}{4}\right) \leq \frac{2}{N^2\beta} \cdot \frac{4}{\eta} \cdot \exp\left(-\frac{N\beta}{4}\right). \end{aligned} \quad (10.37)$$

Suppose \mathcal{L} is the counterclockwise path along the boundary of the rectangle whose vertices are $c - 5\kappa \pm N^{-10}i$ and $d + 5\kappa \pm N^{-10}i$. Suppose \mathcal{L}' consists of the horizontal segments of \mathcal{L} . Suppose

$\eta = \text{Im}z \geq N^{-1}$. Recall that $\tilde{\rho}(x) = \sqrt{(x-c)(d-x)}\tilde{r}(x)\mathbf{1}_{[c,d]}(x)$. According to (10.37) there exists $N_0 = N_0(\epsilon_0, V_p, \kappa) > 0$ such that if $N > N_0$, then

$$\left| \frac{1}{2\pi i} \int_{\mathcal{L}'} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + (2\tilde{m}(\xi) - V_p'(\xi))(\tilde{m}_N(\xi) - \tilde{m}(\xi)) + A(\xi) + B(\xi)}{\tilde{r}(\xi)(z - \xi)} d\xi \right| \leq W_0 N^9 \exp(-\frac{\beta N}{4})$$

where

$$\begin{aligned} A(\xi) &= \frac{1}{N^2} \text{Var}_{\tilde{\mu}}\left(\sum \frac{1}{\xi - \lambda_i}\right) - \frac{1}{N} \left(\frac{2}{\beta} - 1\right) \tilde{m}'_N(\xi) + (1 - c_N) V_p'(\xi) \tilde{m}_N(\xi) \\ B(\xi) &= \int_a^b \frac{V_p'(\xi) - V_p'(t)}{\xi - t} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) dt + (c_N - 1) \int_a^b \frac{V_p'(\xi) - V_p'(t)}{\xi - t} \tilde{\rho}_1^{(N)}(t) dt \\ W_0 &= \frac{32(d-c+10\kappa)}{\beta\pi} \left(\min_{\text{dist}(\xi, [c,d]) \leq 6\kappa} |\tilde{r}(\xi)| \right)^{-1} > 0. \end{aligned}$$

Notice that if ξ is on the vertical segments of \mathcal{L} and $t \in [a, b] = [c - \frac{\kappa}{2}, d + \frac{\kappa}{2}]$, then $\text{dist}(\xi, t) \geq 4\kappa$. So there exist $N_0 > 0$ and $W_1 > 0$ depending on V_p, ϵ_0 and κ such that if $N > N_0$ and ξ is on the vertical segments of \mathcal{L} , then

$$\left| \frac{1}{2\pi i} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + (2\tilde{m}(\xi) - V_p'(\xi))(\tilde{m}_N(\xi) - \tilde{m}(\xi)) + A(\xi) + B(\xi)}{\tilde{r}(\xi)(z - \xi)} \right| \leq W_1 N.$$

Therefore for $N > N_0(\epsilon_0, V_p, a^*, \epsilon_1, \kappa)$

$$\left| \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + (2\tilde{m}(\xi) - V_p'(\xi))(\tilde{m}_N(\xi) - \tilde{m}(\xi)) + A(\xi) + B(\xi)}{\tilde{r}(\xi)(z - \xi)} d\xi \right| \leq W_0 N^9 \exp(-\frac{\beta N}{4}) + 4W_1 N^{-9}.$$

Both $\xi \mapsto \int_a^b \frac{V_p'(\xi) - V_p'(t)}{\xi - t} \tilde{\rho}(t) dt$ and $\xi \mapsto \int_a^b \frac{V_p'(\xi) - V_p'(t)}{\xi - t} \tilde{\rho}_1^{(N)}(t) dt$ are analytic on a neighborhood of the rectangle with boundary \mathcal{L} , so $B(\xi)$ is also analytic on a neighborhood of that rectangle.

So for $N > N_0(\epsilon_0, V_p, \kappa)$

$$\left| \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + (2\tilde{m}(\xi) - V_p'(\xi))(\tilde{m}_N(\xi) - \tilde{m}(\xi)) + A(\xi)}{\tilde{r}(\xi)(z - \xi)} d\xi \right| \leq W_2 N^{-9}$$

where $W_2 > 0$ depends on W_0 and W_1 . Since

$$\tilde{m}(\xi) - \tilde{m}_N(\xi) = \int_a^b \frac{\tilde{\rho}(t) - \tilde{\rho}_1^{(N)}(t)}{\xi - t} dt = \int_a^b (\tilde{\rho}(t) - \tilde{\rho}_1^{(N)}(t)) \frac{t}{\xi(\xi - t)} dt = O(\xi^{-2}) \quad \text{as } \xi \rightarrow \infty$$

we have $\frac{\sqrt{(c-\xi)(d-\xi)}(\tilde{m}_N(\xi) - \tilde{m}(\xi))}{(z - \xi)} = O(\xi^{-2})$ as $\xi \rightarrow \infty$. Therefore by Lemma 8.1 and Cauchy's integral formula,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{(2\tilde{m}(\xi) - V_p'(\xi))(\tilde{m}_N(\xi) - \tilde{m}(\xi))}{\tilde{r}(\xi)(z - \xi)} d\xi &= \\ \frac{i}{\pi} \int_{\mathcal{L}} \frac{\sqrt{(c-\xi)(d-\xi)}(\tilde{m}_N(\xi) - \tilde{m}(\xi))}{(z - \xi)} d\xi &= -2\sqrt{(c-z)(d-z)}(\tilde{m}_N(z) - \tilde{m}(z)). \end{aligned}$$

In other words, if $N > N_0(\epsilon_0, V_p, \kappa)$ and $\text{Im}z = \eta > N^{-1}$ then

$$\left| 2\sqrt{(c-z)(d-z)}(\tilde{m}_N(z) - \tilde{m}(z)) - \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + A(\xi)}{\tilde{r}(\xi)(z - \xi)} d\xi \right| \leq W_2 N^{-9}. \quad (10.38)$$

Suppose \mathcal{L}_1 is the counterclockwise path along the boundary of the rectangle whose vertices are $c - 3\kappa \pm 3\kappa i$ and $d + 3\kappa \pm 3\kappa i$. Suppose \mathcal{L}_2 is the counterclockwise path along the boundary of the rectangle whose vertices are $c - 4\kappa \pm 4\kappa i$ and $d + 4\kappa \pm 4\kappa i$.

If $z = E + i\eta$ with $E \in (c - 4\kappa, d + 4\kappa)$ and $N^{-1} < \eta < 4\kappa$, then by Cauchy's integral formula

$$\int_{\mathcal{L}} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + A(\xi)}{\tilde{r}(\xi)(z - \xi)} d\xi - \int_{\mathcal{L}_2} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + A(\xi)}{\tilde{r}(\xi)(z - \xi)} d\xi = 2\pi i \frac{(\tilde{m}_N(z) - \tilde{m}(z))^2 + A(z)}{\tilde{r}(z)} \quad (10.39)$$

From (10.38) and (10.39), if $N > N_0(\epsilon_0, V_p, \kappa)$, $E \in (c - 4\kappa, d + 4\kappa)$ and $N^{-1} < \eta < 4\kappa$, then

$$\left| 2\tilde{r}(z)\sqrt{(c-z)(d-z)}(\tilde{m}_N(z) - \tilde{m}(z)) - \frac{\tilde{r}(z)}{2\pi i} \int_{\mathcal{L}_2} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + A(\xi)}{\tilde{r}(\xi)(z - \xi)} d\xi - (\tilde{m}_N(z) - \tilde{m}(z))^2 - A(z) \right| \leq W_2 N^{-9}. \quad (10.40)$$

For $\xi \in \mathcal{L}_1 \cup \mathcal{L}_2$, define smooth function $g_\xi : \mathbb{R} \rightarrow \mathbb{C}$ such that

- $g_\xi(t) = \frac{1}{\xi - t}$ if $t \in [c - \kappa, d + \kappa]$.
- $g_\xi(t) = 0$ if $t \notin [c - 2\kappa, d + 2\kappa]$.
- There exists $C_L = C_L(V_p, \kappa) > 0$ such that $\max(\|g_\xi\|_\infty, \|g'_\xi\|_\infty, \|g''_\xi\|_\infty) < C_L$ for all $\xi \in \mathcal{L}_1 \cup \mathcal{L}_2$.

Remark. It's easy to see that there exists such g_ξ .

Recall that $\epsilon_0 < 0.01$ (see Hypothesis 5.1). By Theorem 9.1, for each $\xi \in \mathcal{L}_1 \cup \mathcal{L}_2$ and $N > N_0(\epsilon_0, V_p, \kappa)$,

$$\begin{aligned} \left| \text{Var}_{\tilde{\mu}} \sum \frac{1}{\xi - \lambda_k} \right| &\leq \mathbb{E}^{\tilde{\mu}} \left(\left| \sum g_\xi(\lambda_k) - \mathbb{E} \sum g_\xi(\lambda_k) \right|^2 \right) \\ &\leq 2\mathbb{E}^{\tilde{\mu}} \left(\left| \sum g_\xi(\lambda_k) - N \int g_\xi(t) \tilde{\rho}(t) dt \right|^2 \right) + 2\mathbb{E}^{\tilde{\mu}} \left(\left| \mathbb{E} \sum g_\xi(\lambda_k) - N \int g_\xi(t) \tilde{\rho}(t) dt \right|^2 \right) \\ &\leq 4N^{0.02} \end{aligned}$$

and thus

$$|A(\xi)| \leq \left| \frac{1}{N^2} \text{Var}_{\mu_N} \left(\sum \frac{1}{\xi - \lambda_i} \right) \right| + \left| \frac{1}{N} \left(\frac{2}{\beta} - 1 \right) \tilde{m}'_N(\xi) \right| + \left| (1 - c_N) V'_p(\xi) \tilde{m}_N(\xi) \right| \leq W_3 \cdot N^{-1+\epsilon_0} \quad (10.41)$$

where W_3 depends on V_p , ϵ_0 and κ .

From (10.40), there exist $W_4 = W_4(V_p, \kappa, \epsilon_0) > 0$ and $N_0 = N_0(\epsilon_0, V_p, \kappa) > 0$ such that if $N > N_0$, $E \in [c - 3.5\kappa, d + 3.5\kappa]$ and $N^{-1} < \eta < 3.5\kappa$, then

$$\left| 2\tilde{r}(z)\sqrt{(c-z)(d-z)}(\tilde{m}_N(z)-\tilde{m}(z))-(\tilde{m}_N(z)-\tilde{m}(z))^2-A(z) \right| \leq W_4 \sup_{\xi \in \mathcal{L}_2} \left(|\tilde{m}_N(\xi)-\tilde{m}(\xi)|^2 \right) + W_4 N^{-1+\epsilon_0}. \quad (10.42)$$

By the above equation and the facts that i). $r(z)$ has no zero inside \mathcal{L}_2 ; ii). $|A(\xi)| \leq W_3 N^{-1+\epsilon_0}$ on \mathcal{L}_1 and iii). $\sup_{\mathcal{L}_2} |\tilde{m}_N(\xi) - \tilde{m}(\xi)| \leq \sup_{\mathcal{L}_1} |\tilde{m}_N(\xi) - \tilde{m}(\xi)|$ (because of the principle of max module)

we have that for $N > N_0(\epsilon_0, V_p, \kappa)$ and $z \in \{E + i\eta \in \mathcal{L}_1 \mid \eta > N^{-1}\}$,

$$|\tilde{m}_N(z) - \tilde{m}(z)| \leq W_5 \sup_{\xi \in \mathcal{L}_1} \left(|\tilde{m}_N(\xi) - \tilde{m}(\xi)|^2 \right) + W_5 N^{-1+\epsilon_0} \quad (10.43)$$

where $W_5 > 0$ depends on V_p , ϵ_0 and κ .

If $z \in \mathcal{L}_1$ with $0 < \text{Im}z \leq N^{-1}$, consider \tilde{z} with same real part as z but $\text{Im}\tilde{z} = \text{Im}z + (1/N)$. It's easy to check that $|\tilde{m}_N(z) - \tilde{m}_N(\tilde{z})| \leq \frac{1}{N} \frac{1}{4\kappa^2}$. This together with (10.43) and the continuity of $\tilde{m}_N(z) - \tilde{m}(z)$ we have that for $N > N_0(\epsilon_0, V_p, \kappa)$ and $z \in \mathcal{L}_1$,

$$|\tilde{m}_N(z) - \tilde{m}(z)| \leq W_5 \sup_{\xi \in \mathcal{L}_1} \left(|\tilde{m}_N(\xi) - \tilde{m}(\xi)|^2 \right) + (W_5 + 1)N^{-1+\epsilon_0}. \quad (10.44)$$

Lemma 10.1. *There is a sequence $\{s_N\}$ depending on V_p , κ and ϵ_0 but independent of the choice of $\{c_N\}$ such that $\sup_{\xi \in \mathcal{L}_1} |\tilde{m}_N(\xi) - \tilde{m}(\xi)| \leq s_N$ and $\lim_{N \rightarrow \infty} s_N = 0$. In other words, $|\tilde{m}_N(\xi) - \tilde{m}(\xi)|$ goes to 0 uniformly on \mathcal{L}_1 with a speed independent of $\{c_N\}$.*

Proof. Lemma 10.1 can be directly induced from Lemma 9.4. \square

From the above lemma and (10.44) we have that when $N > N_0(\epsilon_0, V_p, \kappa)$, $\sup_{\xi \in \mathcal{L}_1} |\tilde{m}_N(\xi) - \tilde{m}(\xi)| \leq 2(W_5 + 1)N^{-1+\epsilon_0}$, thus by the principle of max module,

$$\sup_{\xi \in \mathcal{L}_2} |\tilde{m}_N(\xi) - \tilde{m}(\xi)| \leq 2(W_5 + 1)N^{-1+\epsilon_0}. \quad (10.45)$$

By (10.42), for $N > N_0(\epsilon_0, V_p, \kappa)$, if $E \in [c - 3.5\kappa, d + 3.5\kappa]$ and $N^{-1} < \eta < 3.5\kappa$, then

$$|2\tilde{r}(z)\sqrt{(c-z)(d-z)}(\tilde{m}_N(z)-\tilde{m}(z))-(\tilde{m}_N(z)-\tilde{m}(z))^2-A(z)| \leq W_4(2W_5 + 3)N^{-1+\epsilon_0}. \quad (10.46)$$

Lemma 10.2. *If $\text{Im}z \neq 0$, then $|\text{Im}\tilde{m}(z)| \leq \pi \|\tilde{\rho}\|_\infty$.*

Proof. Suppose $z = E + i\eta$. Without loss of generality suppose $\eta > 0$.

$$|\text{Im}\tilde{m}(z)| = \int \frac{\eta}{(E-t)^2 + \eta^2} \tilde{\rho}(t) dt \leq \|\tilde{\rho}\|_\infty \int \frac{\eta}{(E-t)^2 + \eta^2} dt = \|\tilde{\rho}\|_\infty \pi.$$

\square

By the above lemma, for any z in the upper half plane,

$$\begin{aligned} \left| \frac{1}{N} \tilde{m}'_N(z) \right| &= \frac{1}{N^2} |\mathbb{E}^{\tilde{\mu}} \sum \frac{1}{(z - \lambda_k)^2}| \leq \frac{1}{N\eta} |\operatorname{Im} \tilde{m}_N(z)| \leq \frac{1}{N\eta} |\tilde{m}_N(z) - \tilde{m}(z)| + \frac{1}{N\eta} |\operatorname{Im} \tilde{m}(z)| \\ &\leq \frac{1}{N\eta} |\tilde{m}_N(z) - \tilde{m}(z)| + \frac{\|\tilde{\rho}\|_\infty \pi}{N\eta} \end{aligned}$$

Suppose $\delta_* > 0$. It's easy to check that there is a constant $M_1 = M_1(V_p, \kappa, \epsilon_0, \delta_*) > 0$ such that for every $z = E + i\eta \in \{E + i\eta | E \in [c + \delta_*, d - \delta_*], \eta \in (0, 4\kappa)\}$ and $N \geq 1$,

- $M_1^{-1} < |2r(z)\sqrt{(c-z)(d-z)}| < M_1$
- $|\frac{1}{N}(\frac{2}{\beta} - 1)\tilde{m}'_N(z)| \leq \frac{M_1}{N\eta} |\tilde{m}_N(z) - \tilde{m}(z)| + \frac{M_1}{N\eta}$
- $|(1 - c_N)V'_p(z)\tilde{m}_N(z)| \leq \frac{N^{\epsilon_0}}{N\eta} \cdot M_1$.

Thus by (10.46), for $z \in \{E + i\eta | E \in [c + \delta_*, d - \delta_*], \eta \in (\frac{1}{N}, 3.5\kappa)\}$ and $N > N_0(V_p, \kappa, \epsilon_0, \delta_*)$ the following inequality hold uniformly

$$\begin{aligned} &|\tilde{m}_N(z) - \tilde{m}(z)| \\ &\leq M_1 |\tilde{m}_N(z) - \tilde{m}(z)|^2 + M_1 \frac{1}{N^2} |\operatorname{Var}_{\tilde{\mu}} \sum \frac{1}{z - \lambda_k}| + M_1^2 \frac{1}{N\eta} |\tilde{m}_N(z) - \tilde{m}(z)| + M_1^2 \frac{1 + N^{\epsilon_0}}{N\eta} + M_1 W_4(2W_5 + 3) \frac{N^{\epsilon_0}}{N} \end{aligned} \quad (10.47)$$

Suppose $p_0 > 0$ and $\Omega_* = \{E + i\eta | E \in [c + \delta_*, d - \delta_*], \eta \in (N^{-1+\epsilon_0+p_0}, 3.5\kappa)\}$. Notice that $\frac{1 + N^{\epsilon_0}}{N\eta}$ converges to 0 uniformly on Ω_* since $\eta > N^{-1+\epsilon_0+p_0}$. We have the following lemma.

Lemma 10.3. *Suppose $\{t_N\}$ is a sequence of positive numbers converging to 0. Also suppose that $\{t_N\}$ depends on V_p, κ and ϵ_0 but is independent of the choice of $\{c_N\}$. Suppose $\frac{1}{N^2} |\operatorname{Var}_{\tilde{\mu}} \sum \frac{1}{z - \lambda_k}| \leq t_N$ for all $z \in \Omega_*$ and $N \geq 1$, i.e., converges to 0 uniformly on Ω_* with speed independent of $\{c_N\}$. Then there exist*

- a constant $C_2 > 0$ depending on V_p, κ, ϵ_0 and δ_*
- a sequence of positive numbers $\{\epsilon_N\}$ depending on $V_p, \kappa, \epsilon_0, \delta_*, \{t_N\}$ and p_0 st. $\lim_{N \rightarrow \infty} \epsilon_N = 0$
- a constant $N_0 > 0$ depending on $V_p, \kappa, \epsilon_0, \delta_*, p_0$

such that when $N > N_0$

$$|\tilde{m}_N(z) - \tilde{m}(z)| \leq C_2 |\tilde{m}_N(z) - \tilde{m}(z)|^2 + \epsilon_N \quad \forall z \in \Omega_*$$

and thus

$$|\tilde{m}_N(z) - \tilde{m}(z)| \geq \frac{1}{C_2} - 2\epsilon_N \quad \text{or} \quad |\tilde{m}_N(z) - \tilde{m}(z)| \leq 2\epsilon_N \quad \forall z \in \Omega_*$$

The above lemma, Lemma 10.1 and the continuity of $\tilde{m}_N - \tilde{m}$ leads to the fact that if $\frac{1}{N^2} |\operatorname{Var}_{\mu_N} \sum \frac{1}{z - \lambda_k}| \leq t_N$ for all $z \in \Omega_*$, then $|\tilde{m}_N(z) - \tilde{m}(z)| \leq 2\epsilon_N$ for all $z \in \Omega_*$. This together with (10.47) give the following theorem.

Theorem 10.4. Suppose $0 < \delta_* < \frac{d-c}{2}$, $p_0 > 0$ and $\Omega_* = \{E + i\eta \mid E \in (c + \delta_*, d - \delta_*), \eta \in (N^{-1+\epsilon_0+p_0}, 3.5\kappa)\}$. Suppose $\{t_N\}$ is a sequence of positive numbers converging to 0. Also suppose that $\{t_N\}$ depends on V_p, κ and ϵ_0 . Suppose $\frac{1}{N^2} |\text{Var}_{\tilde{\mu}} \sum \frac{1}{z - \lambda_k}| \leq t_N$ for all $N \geq 1$ and $z \in \Omega_*$, i.e., converges to 0 uniformly on Ω_* with speed independent of $\{c_N\}$. Then there exist

- a constant $M_2 > 0$ depending on V_p, κ, ϵ_0 and δ_*
- a constant $N_0 > 0$ depending on $V_p, \kappa, \epsilon_0, \delta_*, \{t_N\}$ and p_0

such that when $N > N_0$

$$|\tilde{m}_N(z) - \tilde{m}(z)| \leq M_2 \left(\frac{1}{N^2} |\text{Var}_{\tilde{\mu}} \sum \frac{1}{z - \lambda_k}| + \frac{N^{\epsilon_0}}{N\eta} \right) \quad \forall z \in \Omega_*$$

Corollary 10.5. There are $N_0 = N_0(\epsilon_0, V_p, \kappa) > 0$, $W_8 = W_8(V_p, \kappa, \epsilon_0) > 0$ such that if $N > N_0$ $z = E + i\eta$ with $E \leq a - 2$ and $\eta > N^{-1}$, then $|m_N(z) - m(z)| \leq W_8 N^{-1+\epsilon_0}$.

Proof. Suppose $z = E + i\eta$ with $E \leq a - 2$ and $\eta > N^{-1}$. According to (10.38), when $N > N_0(\epsilon_0, V_p, \kappa)$,

$$|2\sqrt{(c-z)(d-z)}(\tilde{m}_N(z) - \tilde{m}(z)) - \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + A(\xi)}{r(\xi)(z - \xi)} d\xi| \leq W_2 N^{-9}$$

where $W_2 = W_2(V_p, \kappa, \epsilon_0) > 0$. Since $E \leq a - 2$ we have

$$\int_{\mathcal{L}} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + A(\xi)}{r(\xi)(z - \xi)} d\xi = \int_{\mathcal{L}_2} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + A(\xi)}{r(\xi)(z - \xi)} d\xi$$

and $2\sqrt{(c-z)(d-z)} \geq 4$, thus

$$|\tilde{m}_N(z) - \tilde{m}(z)| \leq \frac{1}{8\pi} \left| \int_{\mathcal{L}_2} \frac{(\tilde{m}_N(\xi) - \tilde{m}(\xi))^2 + A(\xi)}{r(\xi)(z - \xi)} d\xi \right| + \frac{W_2}{4} N^{-9}.$$

Because of (10.41) and $|z - \xi| \geq 1 (\forall \xi \in \mathcal{L}_2)$, we have that for $N > N_0(\epsilon_0, V_p, \kappa)$,

$$\begin{aligned} |\tilde{m}_N(z) - \tilde{m}(z)| &\leq \frac{1}{8\pi} \left| \int_{\mathcal{L}_2} \frac{|\tilde{m}_N(\xi) - \tilde{m}(\xi)|^2 + W_3 N^{-1+\epsilon_0}}{|\tilde{r}(\xi)|} d\xi \right| + \frac{W_2}{4} N^{-9} \\ &\leq \frac{1}{8\pi} \left(\sup_{\xi \in \mathcal{L}_2} (|\tilde{m}_N(\xi) - \tilde{m}(\xi)|^2) + W_3 N^{-1+\epsilon_0} \right) \left(\min_{\text{dist}(\xi, [c, d]) \leq 6\kappa} |r(\xi)| \right)^{-1} 2(d - c + 16\kappa) + \frac{W_2}{4} N^{-10} \\ &\leq W_6 \sup_{\xi \in \mathcal{L}_2} (|\tilde{m}_N(\xi) - \tilde{m}(\xi)|^2) + W_7 N^{-1+\epsilon_0} \end{aligned}$$

where W_6 and W_7 depend on V_p, κ and ϵ_0 . According to (10.45), when $N > N_0(\epsilon_0, V_p, a^*, \epsilon_1, \kappa)$,

$$|\tilde{m}_N(z) - \tilde{m}(z)| \leq 2W_6(W_5 + 1)N^{-2+2\epsilon_0} + W_7 N^{-1+\epsilon_0} \leq W_8 N^{-1+\epsilon_0}$$

and W_8 depends on V_p, κ and ϵ_0 . □

11 Analysis on $\tilde{\mathcal{H}}$

According to Hypothesis 5.1, there exists $U_p > 0$ such that $V_p'(x) \geq -2U_p$ for all x in a neighbor of $[a, b]$. Recall that $\gamma_k = \gamma_k(N)$ and $\tilde{\gamma}_k = \tilde{\gamma}_k(N)$ are defined as

$$\int_c^{\gamma_k} \tilde{\rho}(t) dt = \frac{k}{N} \quad \int_c^{\tilde{\gamma}_k} \tilde{\rho}(t) dt = \frac{k - 1/2}{N}.$$

Lemma 11.1. • If u_1, v_1 are both in $[0, +\infty)$, then $|u_1^{3/2} - v_1^{3/2}| \geq \frac{1}{2}|u_1 - v_1|^{3/2}$.

• If u_2, v_2 are both in $[0, 1]$, then $\frac{1}{2}(u_2^{2/3} + v_2^{2/3}) \leq (u_2 + v_2)^{2/3} \leq 2(u_2^{2/3} + v_2^{2/3})$.

• If $0 \leq u_3 \leq v_3$, then $\frac{1}{2}(v_3 - u_3)(\sqrt{v_3} + \sqrt{u_3}) \leq v_3^{3/2} - u_3^{3/2} \leq (v_3 - u_3)(\sqrt{v_3} + \sqrt{u_3})$.

Proof. This lemma is trivial. \square

From Lemma 5.1 we have:

Lemma 11.2. There exist $p_1 > 0, p_2 > 0$ depending on V_p, κ, ϵ_0 but independent of $\{c_N\}$ such that if $N \geq 2$, then

$$\begin{aligned} p_1 \sqrt{t - c} \leq \tilde{\rho}(t) \leq p_2 \sqrt{t - c} & \quad \forall t \in [c, \tilde{\gamma}_{\frac{2}{3}N}], \\ p_1 \sqrt{d - t} \leq \tilde{\rho}(t) \leq p_2 \sqrt{d - t} & \quad \forall t \in [\tilde{\gamma}_{\frac{1}{3}N}, d]. \end{aligned}$$

Theorem 7.1 with $h \equiv 0$ gives the following lemma.

Lemma 11.3 (Initial estimation). For any $A_1 > 0$ there exist $A_2 > 0, N_1 > 0$ both depending on V_p, κ and ϵ_0 and A_1 but independent of the choice of $\{c_N\}$ such that if $N > N_1$ then

$$\mathbb{P}^{\tilde{\mu}_s}(\exists k \in [1, N] \text{ st. } |\lambda_k - \gamma_k| > A_1) \leq \exp(-A_2 N).$$

Corollary 11.4. For any $A_1 > 0$ there exist $A_2 > 0, N_1 > 0$ both depending on V_p, κ, ϵ_0 and A_1 but independent of the choice of $\{c_N\}$ such that if $N > N_1$ then

$$\mathbb{P}^{\tilde{\mu}_s}(\exists k \in [1, N] \text{ st. } |\lambda_k - \tilde{\gamma}_k| > A_1) \leq \exp(-A_2 N).$$

Proof of Corollary 11.4. Suppose $j \leq \frac{N}{2}$. By Lemma 11.2 and the fact that $|x_1^{3/2} - x_2^{3/2}| \geq \frac{1}{2}|x_1 - x_2|^{3/2}$ for nonnegative x_1, x_2 ,

$$\frac{1}{2N} = \int_{\tilde{\gamma}_j}^{\gamma_j} \tilde{\rho}(t) dt \geq p_1 \int_{\tilde{\gamma}_j}^{\gamma_j} \sqrt{t - c} dt = \frac{2p_1}{3} |(\tilde{\gamma}_j - c)^{3/2} - (\gamma_j - c)^{3/2}| \geq \frac{p_1}{3} |\tilde{\gamma}_j - \gamma_j|^{3/2}.$$

Therefore $|\tilde{\gamma}_j - \gamma_j| \leq \left(\frac{3}{2p_1}\right)^{2/3} N^{-2/3}$. By the same argument we have this inequality for $j \geq \frac{N}{2}$. This together with Lemma 11.3 completes the proof. \square

Recall that $\tilde{\mu}$ can be written as

$$\frac{1}{Z_{\tilde{\mu}}} e^{-\beta N \mathcal{H}_{\tilde{\mu}}} \prod_{i=1}^N \mathbb{1}_{[a, b]}(\lambda_i)$$

where $\mathcal{H}_{\tilde{\mu}} = \frac{1}{2} \sum c_N V_p(\lambda_i) - \frac{1}{N} \sum_{i < j} \ln |\lambda_i - \lambda_j|$.

Set

$$\tilde{\mathcal{H}} = \mathcal{H}_{\tilde{\mu}} + M \sum_{\alpha=1}^l X_{\alpha}^2 \quad \text{where} \quad X_{\alpha} = N^{-1/2} \sum_{i=1}^N (g_{\alpha}(\lambda_i) - g_{\alpha}(\tilde{\gamma}_i)).$$

Here g_{α} is an N -independent function such that

- $\|g_{\alpha}\|_{\infty} + \|g'_{\alpha}\|_{\infty} + \|g''_{\alpha}\|_{\infty} < \infty$.
- $g'_{\alpha}(\tilde{\gamma}_k) = \sqrt{2} \cos\left(2\pi(k - \frac{1}{2})\frac{\alpha}{2N}\right)$

Remark. The g_{α} satisfying above conditions exists. For example, we can set $g_{\alpha}(x)$ to be a C^2 function with

$$g_{\alpha}(x) = \begin{cases} \int_c^x \sqrt{2} \cos(\pi \alpha \int_c^t \tilde{\rho}(s) ds) dt & \text{if } x \in [c, d]; \\ 0 & \text{if } x \notin [c-1, d+1]. \end{cases}$$

Remark. $\tilde{\mathcal{H}}$ depends on $V_p, \kappa, \epsilon_0, c_N, M, l$ and N .

Set $\mathbf{G}_{\alpha} = N^{-1/2} (g'_{\alpha}(\tilde{\gamma}_1), \dots, g'_{\alpha}(\tilde{\gamma}_N))$ and $\Delta = \max\left(\frac{1}{N} \sum |\lambda_j - \tilde{\gamma}_j|, \frac{1}{N} \sum (\lambda_j - \tilde{\gamma}_j)^2\right)$. By direct computation, the l^2 norm of \mathbf{G} is 1. Also by direct computation, for any $\mathbf{v} \in \mathbb{R}^N$,

$$\begin{aligned} \langle \mathbf{v}, (\nabla^2 \tilde{\mathcal{H}}) \mathbf{v} \rangle &= \frac{1}{N} \sum_{i < j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} + \frac{1}{2} c_N \sum V_p''(\lambda_i) v_i^2 + 2M \sum_{\alpha=1}^l \left[\left(\frac{1}{\sqrt{N}} \sum g'_{\alpha}(\lambda_j) v_j \right)^2 + X_{\alpha} \sum \frac{1}{\sqrt{N}} g''_{\alpha}(\lambda_j) v_j^2 \right] \\ &\geq \frac{1}{N} \sum_{i < j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} + M \sum_{\alpha=1}^l |\langle \mathbf{G}_{\alpha}, \mathbf{v} \rangle|^2 - \left[c_N W + 2\Delta M \sum_{\alpha=1}^l (\|g''_{\alpha}\|_{\infty}^2 + \|g'_{\alpha}\|_{\infty} \|g''_{\alpha}\|_{\infty}) \right] \|\mathbf{v}\|^2. \end{aligned} \tag{11.48}$$

In the last inequality we used the fact that

$$\begin{aligned} 2 \left(\sum g'_{\alpha}(\lambda_j) v_j \right)^2 &\geq \left(\sum g'_{\alpha}(\tilde{\gamma}_j) v_j \right)^2 - 2 \left(\sum [g'_{\alpha}(\lambda_j) - g'_{\alpha}(\tilde{\gamma}_j)] v_j \right)^2 \\ &\geq \left(\sum g'_{\alpha}(\tilde{\gamma}_j) v_j \right)^2 - 2 \sum [g'_{\alpha}(\lambda_j) - g'_{\alpha}(\tilde{\gamma}_j)]^2 \|\mathbf{v}\|^2. \end{aligned}$$

Remark. The above lower bound of $\langle \mathbf{v}, (\nabla^2 \tilde{\mathcal{H}}) \mathbf{v} \rangle$ was first given in [7].

For $n \in \mathbb{N}$ define $m(n) \in [-N+1, N]$ and $m(n) \equiv n \pmod{2N}$. Suppose $\epsilon > 0$. For $k, l \in [-N+1, N]$, define

$$R_{k,l} = R_{k,l}(\epsilon) = \frac{1}{N} \frac{\epsilon^{2/3}}{\epsilon^2 + \frac{d(k,l)^2}{N^2}}$$

where $d(k, l) = |m(k) - m(l)|$.

Suppose $i, j \in [1, N]$. Define

$$Q_{i,j} = Q_{i,j}(\epsilon) = R_{i,j} + R_{i,1-j} + R_{1-i,j} + R_{1-i,1-j}.$$

The following proposition was first given in [7].

Proposition 11.1. *There exists $\tilde{c}_1 > 0$ depending on V_p, κ, ϵ_0 but independent of $\{c_N\}$ satisfying the following property. For any $0 < \epsilon < 0.1$ there exist $\tilde{c}_2 > 0, N_0 > 0$ depending on V_p, κ, ϵ_0 and ϵ but independent of $\{c_N\}$ such that if $N > N_0$ and $i, j \in [1, N]$, then*

$$\mathbb{P}^{\tilde{\mu}_s} \left(\frac{1}{N} \frac{1}{(\lambda_i - \lambda_j)^2} \leq \tilde{c}_1 Q_{i,j} \right) \leq \exp(-\tilde{c}_2 N).$$

Proof of Proposition 11.1. According to Lemma 11.2, when $N \geq 2$,

$$\begin{aligned} \left(\frac{3}{4p_2}\right)^{2/3} \left(\frac{k}{N}\right)^{2/3} &\leq \tilde{\gamma}_k - c \leq \left(\frac{3}{2p_2}\right)^{2/3} \left(\frac{k}{N}\right)^{2/3} & \text{if } k \leq \frac{2N}{3} \\ \left(\frac{3}{2p_2}\right)^{2/3} \left(\frac{N-k+1/2}{N}\right)^{2/3} &\leq d - \tilde{\gamma}_k \leq \left(\frac{3}{2p_2}\right)^{2/3} \left(\frac{N-k+1/2}{N}\right)^{2/3} & \text{if } k \geq \frac{N}{3} \end{aligned}$$

- Case 1: $|i - j| \leq \epsilon N$.

In this case, we have $\{i, j\} \subset [1, \frac{2}{3}N]$ or $\{i, j\} \subset [\frac{1}{3}N, N]$ since $\epsilon < 0.1$. Without loss of generality suppose $i \geq j$.

If $\{i, j\} \subset [1, \frac{2}{3}N]$, then by Lemma 11.2, $\epsilon \geq \int_{\tilde{\gamma}_j}^{\tilde{\gamma}_i} \rho(t) dt \geq \frac{p_1}{3} |\tilde{\gamma}_1 - \tilde{\gamma}_j|^{3/2}$. If $\{i, j\} \subset [\frac{1}{3}N, N]$, by Lemma 11.2 we have the same estimation. Thus $|\tilde{\gamma}_i - \tilde{\gamma}_j| \leq \left(\frac{3}{p_1}\right)^{2/3} \epsilon^{2/3}$.

- Case 2: $|i - j| \geq \epsilon N$ and $\{i, j\} \subset [1, \frac{2}{3}N]$.

Without loss of generality suppose $i \geq j$. If $t \in [\tilde{\gamma}_{\frac{i+j}{2}}, \tilde{\gamma}_i]$, then

$$\tilde{\rho}(t) \geq p_1 \sqrt{t - c} \geq p_1 \sqrt{\tilde{\gamma}_{\frac{i+j}{2}} - c} \geq p_1 \sqrt{\left(\frac{3}{4p_2}\right)^{2/3} \left(\frac{(i+j)/2}{N}\right)^{2/3}} \geq p_1 \left(\frac{3}{4p_2}\right)^{1/3} \left(\frac{\epsilon}{2}\right)^{1/3}$$

thus

$$\int_{\tilde{\gamma}_j}^{\tilde{\gamma}_i} \tilde{\rho}(t) dt \geq \int_{\tilde{\gamma}_{\frac{i+j}{2}}}^{\tilde{\gamma}_i} \tilde{\rho}(t) dt \geq |\tilde{\gamma}_i - \tilde{\gamma}_{\frac{i+j}{2}}| p_1 \left(\frac{3}{4p_2}\right)^{1/3} \left(\frac{\epsilon}{2}\right)^{1/3}. \quad (11.49)$$

According to Lemma 11.2 and Lemma 11.1

$$\begin{aligned} \frac{i-j}{N} &= \int_{\tilde{\gamma}_j}^{\tilde{\gamma}_i} \tilde{\rho}(t) dt \geq p_1 \int_{\tilde{\gamma}_j}^{\tilde{\gamma}_i} \sqrt{t - c} dt = \frac{2p_1}{3} |(\tilde{\gamma}_i - c)^{3/2} - (\tilde{\gamma}_j - c)^{3/2}| \geq \frac{p_1}{3} (\tilde{\gamma}_i - \tilde{\gamma}_j) (\sqrt{\tilde{\gamma}_i - c} + \sqrt{\tilde{\gamma}_j - c}); \\ \frac{i-j}{2N} &= \int_{\tilde{\gamma}_{\frac{i+j}{2}}}^{\tilde{\gamma}_i} \tilde{\rho}(t) dt \leq p_2 \int_{\tilde{\gamma}_{\frac{i+j}{2}}}^{\tilde{\gamma}_i} \sqrt{t - c} dt = \frac{2p_2}{3} |(\tilde{\gamma}_i - c)^{3/2} - (\tilde{\gamma}_{\frac{i+j}{2}} - c)^{3/2}| \leq \frac{2p_2}{3} (\tilde{\gamma}_i - \tilde{\gamma}_{\frac{i+j}{2}}) (\sqrt{\tilde{\gamma}_i - c} + \sqrt{\tilde{\gamma}_{\frac{i+j}{2}} - c}) \end{aligned}$$

therefore

$$\frac{\tilde{\gamma}_i - \tilde{\gamma}_j}{\tilde{\gamma}_i - \tilde{\gamma}_{\frac{i+j}{2}}} \leq \frac{\frac{i-j}{N} \frac{3}{p_1 (\sqrt{\tilde{\gamma}_i - c} + \sqrt{\tilde{\gamma}_j - c})}}{\frac{i-j}{2N} \frac{3}{2p_2 (\sqrt{\tilde{\gamma}_i - c} + \sqrt{\tilde{\gamma}_{\frac{i+j}{2}} - c})}} = \frac{4p_2}{p_1} \frac{\sqrt{\tilde{\gamma}_i - c} + \sqrt{\tilde{\gamma}_{\frac{i+j}{2}} - c}}{\sqrt{\tilde{\gamma}_i - c} + \sqrt{\tilde{\gamma}_j - c}} \leq \frac{8p_2}{p_1}.$$

This together with (11.49) gives

$$\frac{|i-j|}{N} = \int_{\tilde{\gamma}_j}^{\tilde{\gamma}_i} \tilde{\rho}(t) dt \geq |\tilde{\gamma}_i - \tilde{\gamma}_j| \frac{p_1^2}{8p_2} \left(\frac{3}{4p_2}\right)^{1/3} \left(\frac{\epsilon}{2}\right)^{1/3}$$

and

$$|\tilde{\gamma}_i - \tilde{\gamma}_j| \leq \frac{|i-j|}{N\epsilon^{1/3}} \frac{8p_2}{p_1^2} \left(\frac{8p_2}{3}\right)^{1/3}. \quad (11.50)$$

- Case 3: $|i-j| \geq \epsilon N$ and $\{i, j\} \subset [\frac{N}{3}, N]$.

By the same argument as Case 2 we also have the estimation (11.50) in this case.

- Case 4: $|i-j| \geq \epsilon N$ and one of i, j is in $[\frac{N}{3}, N]$ while the other one is in $[1, \frac{2N}{3}]$.

Without loss of generality suppose $i > j$. Recall that $\epsilon < 0.1$ and we have

$$\frac{|i-j|}{N} = \int_{\tilde{\gamma}_j}^{\tilde{\gamma}_i} \tilde{\rho}(t) dt \geq \frac{1}{3} \geq |\tilde{\gamma}_i - \tilde{\gamma}_j| \frac{1}{3(d-c)} \geq |\tilde{\gamma}_i - \tilde{\gamma}_j| \frac{1}{3(d-c)} \epsilon^{1/3}$$

and

$$|\tilde{\gamma}_i - \tilde{\gamma}_j| \leq \frac{|i-j|}{N\epsilon^{1/3}} 3(d-c).$$

From Case 1-4, if $N \geq 2$ and i, j in $[1, N]$, then

$$|\tilde{\gamma}_i - \tilde{\gamma}_j| \leq C_4 \epsilon^{2/3} + C_4 \frac{|i-j|}{N\epsilon^{1/3}} \quad (11.51)$$

where $C_4 = \max\left(\left(\frac{3}{p_1}\right)^{2/3}, \frac{8p_2}{p_1^2} \left(\frac{8p_2}{3}\right)^{1/3}, 3(d-c)\right)$ depending on V_p, κ and ϵ_0 .

If $i \geq \frac{1}{3}N$, then

$$\frac{1}{2N} = \int_{\tilde{\gamma}_i}^{\gamma_i} \tilde{\rho}(t) dt \geq p_1 \int_{\tilde{\gamma}_i}^{\gamma_i} \sqrt{d-t} dt = \frac{2}{3} p_1 \left((d-\tilde{\gamma}_i)^{3/2} - (d-\gamma_i)^{3/2} \right) \geq \frac{p_1}{3} |\tilde{\gamma}_i - \gamma_i|^{3/2}.$$

Suppose $N \geq 3$. If $i \leq 0.5N$, then $\gamma_i \leq \tilde{\gamma}_{2N/3}$ and

$$\frac{1}{2N} = \int_{\tilde{\gamma}_i}^{\gamma_i} \tilde{\rho}(t) dt \geq p_1 \int_{\tilde{\gamma}_i}^{\gamma_i} \sqrt{t-c} dt = \frac{2}{3} p_1 \left| (\tilde{\gamma}_i - c)^{3/2} - (\gamma_i - c)^{3/2} \right| \geq \frac{p_1}{3} |\tilde{\gamma}_i - \gamma_i|^{3/2}.$$

So for $N \geq 3$ and $1 \leq i \leq N$,

$$|\tilde{\gamma}_i - \gamma_i| \leq \left(\frac{3}{2p_1 N} \right)^{2/3} \quad (11.52)$$

From Lemma 11.3, (11.51) and (11.52), there exist $A_2 > 0$, $N_1 > 0$ depending on $\epsilon, V_p, \kappa, \epsilon_0$ such that when $N > N_1$,

$$\mathbb{P}^{\tilde{\mu}_s} \left(\exists i, j \in [1, N] \text{ st. } |\lambda_i - \lambda_j| > (1 + C_4) \epsilon^{2/3} + C_4 \frac{|i-j|}{N\epsilon^{1/3}} \right) \leq \exp(-A_2 N).$$

So when $N > N_1$ and $i, j \in [1, N]$,

$$\mathbb{P}^{\tilde{\mu}_s} \left(|\lambda_i - \lambda_j|^{-2} \leq \frac{C_5 \epsilon^{2/3}}{\epsilon^2 + \frac{|i-j|^2}{N^2}} \right) \leq \exp(-A_2 N)$$

where $C_5 = \min\left(\frac{1}{2C_4^2}, \frac{1}{2(1+C_4)^2}\right)$ depending on V_p, κ and ϵ_0 .

The proof will be complete if we can show that for $N \geq 1$ and i, j in $[1, N]$,

$$|i - j| \leq \min(d(i, j), d(1 - i, j), d(i, 1 - j), d(1 - i, 1 - j)).$$

This can be proved by exactly the same argument as the last paragraph in the proof of Lemma 3.1 of [7]. \square

12 Convexification of Hamiltonian

In this section we use the method developed in [7] to convexify the Hamiltonian $\mathcal{H}_{\tilde{\mu}}$.

Suppose \mathcal{Q} is a quadratic form on \mathbb{C}^N such that

$$\langle \mathbf{v}, \mathcal{Q} \mathbf{v} \rangle = \sum_{i,j=1}^N Q_{i,j} (v_i - v_j)^2.$$

So \mathcal{Q} depends on ϵ . The following Lemma about \mathcal{Q} is Lemma 3.3 of [7]. It does not depend on the β ensemble model.

Lemma 12.1. *For any $M > 0$, there exist $0 < \epsilon_M < 0.1$, $l_M > 0$ and $N_0 > 0$ all depending on M such that when $0 < \epsilon < \epsilon_M$, $l > l_M$ and $N > N_0$, we have the following inequality between operators on \mathbb{R}^N :*

$$\mathcal{Q} + M \sum_{\alpha=1}^l |\mathbf{G}_\alpha\rangle\langle \mathbf{G}_\alpha| \geq M.$$

Proof. See [7]. \square

Suppose $\theta(x) = (x - 1)^2 \mathbb{1}_{x>1} + (x + 1)^2 \mathbb{1}_{x<-1}$. Set

$$\mathcal{H}_\nu = \tilde{\mathcal{H}} + \psi^{(s)} + \sum_{i,j} \psi_{i,j} = \mathcal{H}_{\tilde{\mu}} + \psi^{(s)} + \sum_{i,j} \psi_{i,j} + M \sum_{\alpha=1}^l X_\alpha^2$$

where

- $X_\alpha = N^{-1/2} \sum_{j=1}^N (g_\alpha(\lambda_j) - g_\alpha(\tilde{\gamma}_j))$;
- $\psi^{(s)}(\lambda) = N\theta\left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2\right)$;
- $\psi_{i,j}(\lambda) = \frac{1}{N}\theta\left(\sqrt{\tilde{c}_1 N Q_{i,j}}(\lambda_i - \lambda_j)\right)$

where \tilde{c}_1 and $Q_{i,j}$ are defined in Section 11. \tilde{c}_1 depends on V_p, ϵ_0 and κ . $Q_{i,j}$ depends on i, j, ϵ and N . Therefore \mathcal{H}_ν is a random variable depending on $V_p, \kappa, \epsilon_0, c_N, \tilde{c}_1, M, l, s, \epsilon$ and N .

Lemma 12.2. *For any $C_{**} > 0$, there exist $M > 0$, $l > 0$, $s > 0$, $0 < \epsilon < 0.1$, $N_0 > 0$ depending on C_{**} and \tilde{c}_1 such that if $N > N_0$, then*

$$1. \langle \mathbf{v}, (\nabla^2 \mathcal{H}_\nu) \mathbf{v} \rangle \geq C_{**} \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbb{R}^N, \lambda \in [a, b]^N,$$

2. $\mathcal{Q} + \frac{4M}{c_1} \sum_{\alpha=1}^l |\mathbf{G}_\alpha\rangle\langle\mathbf{G}_\alpha| \geq \frac{4M}{c_1}$ holds as inequality between operators on \mathbb{R}^N ,

3. For any $\forall \lambda \in [a, b]^N$, $(1 - C(M, l)\Delta) + 4s(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 - 1)\mathbf{1}_{(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 > 1)} > 0$ where

$$C(M, l) = 2M \sum_{\alpha=1}^l (\|g''_\alpha\|_\infty^2 + \|g'_\alpha\|_\infty \|g''_\alpha\|_\infty).$$

To prove Lemma 12.2 we need the following technical lemma.

Lemma 12.3. For any $w_1 > 0$, there exist $w_2 > 0$, $w_3 > 0$ such that

$$(1 - w_1 \max(x, \sqrt{x})) + (4w_3(w_3x - 1)\mathbf{1}_{(w_3x-1>0)}) \geq w_2 \quad \forall x \geq 0.$$

Proof. Set $w_3 = 1 + \max(w_1, w_1^2, \frac{\sqrt{w_1}}{2}, (\frac{w_1}{4})^{2/3})$ and $f(x) = (1 - w_1 \max(x, \sqrt{x})) + (4w_3(w_3x - 1)\mathbf{1}_{(w_3x-1>0)})$.

• If $x \geq 1$, then $x > \frac{1}{w_3}$ and

$$\begin{aligned} f(x) &= 1 - w_1x + 4w_3(w_3x - 1) = (4w_3^2 - w_1)x + 1 - 4w_3 \geq (4w_3^2 - w_1)\frac{1}{w_3} + 1 - 4w_3 \quad (\text{since } 4w_3^2 - w_1 > 0) \\ &= 1 - \frac{w_1}{w_3} \geq 1 - \frac{w_1}{1 + w_1} = \frac{1}{1 + w_1}. \end{aligned}$$

• If $0 \leq x \leq \frac{1}{w_3}$, then $f(x) = 1 - w_1\sqrt{x} \geq 1 - \frac{w_1}{\sqrt{w_3}} \geq 1 - \frac{w_1}{\sqrt{1+w_1^2}}$.

• If $\frac{1}{w_3} < x < 1$, then

$$\begin{aligned} f(x) &= 1 - w_1\sqrt{x} + 4w_3(w_3x - 1) = 1 + (4w_3^2\sqrt{x} - w_1)\sqrt{x} - 4w_3 \\ &> 1 + (4w_3^2\frac{1}{\sqrt{w_3}} - w_1)\frac{1}{\sqrt{w_3}} - 4w_3 \quad (\text{since } 4w_3^2\frac{1}{\sqrt{w_3}} - w_1 > 0) \\ &= 1 - \frac{w_1}{\sqrt{w_3}} \geq 1 - \frac{w_1}{\sqrt{1+w_1^2}}. \end{aligned}$$

Setting $w_2 = \min(\frac{1}{1+w_1}, 1 - \frac{w_1}{\sqrt{1+w_1^2}}) > 0$ we complete the proof. \square

Proof of Lemma 12.2. Notice that $|c_N| \leq 2$. By (11.48), for any $N \geq 1$ and $\mathbf{v} \in \mathbb{R}^N$,

$$\begin{aligned} \langle \mathbf{v}, (\nabla^2 \mathcal{H}_\nu) \mathbf{v} \rangle &\geq \frac{1}{N} \sum_{i < j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} + M \sum_{\alpha=1}^l |\langle \mathbf{G}_\alpha, \mathbf{v} \rangle|^2 - \left[2W + 2\Delta M \sum_{\alpha=1}^l (\|g''_\alpha\|_\infty^2 + \|g'_\alpha\|_\infty \|g''_\alpha\|_\infty) \right] \|\mathbf{v}\|^2 \\ &\quad + \tilde{c}_1 \sum_{i,j} Q_{i,j} \theta''(\sqrt{\tilde{c}_1 N Q_{i,j}} (\lambda_i - \lambda_j)) (v_i - v_j)^2 + 2s\theta'(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2) \|\mathbf{v}\|^2 \\ &\quad + \frac{(2s)^2}{N} \theta''\left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2\right) \left(\sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i) v_i\right)^2. \end{aligned}$$

For $N \geq 1$ and $\mathbf{v} \in \mathbb{R}^N$,

$$\frac{1}{N} \sum_{i < j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} + \tilde{c}_1 \sum_{i,j} Q_{i,j} \theta''(\sqrt{\tilde{c}_1 N Q_{i,j}}(\lambda_i - \lambda_j))(v_i - v_j)^2 \quad (12.53)$$

$$= \sum_{i \neq j} \left[\frac{1}{2N} \frac{1}{(\lambda_i - \lambda_j)^2} + \tilde{c}_1 Q_{i,j} \theta''(\sqrt{\tilde{c}_1 N Q_{i,j}}(\lambda_i - \lambda_j)) \right] (v_i - v_j)^2 \quad (12.54)$$

$$\geq \sum_{i \neq j} \left[\frac{1}{2N} \frac{1}{(\lambda_i - \lambda_j)^2} + \tilde{c}_1 Q_{i,j} \mathbf{1}_{\left(\frac{1}{(\lambda_i - \lambda_j)^2} < \tilde{c}_1 N Q_{i,j}\right)} \right] (v_i - v_j)^2 \quad (12.55)$$

$$\geq \sum_{i \neq j} \frac{1}{2} \tilde{c}_1 Q_{i,j} (v_i - v_j)^2. \quad (12.56)$$

Therefore

$$\begin{aligned} \langle \mathbf{v}, (\nabla^2 \mathcal{H}_\nu) \mathbf{v} \rangle &\geq \sum_{i \neq j} \frac{1}{2} \tilde{c}_1 Q_{i,j} (v_i - v_j)^2 + M \sum_{\alpha=1}^l |\langle \mathbf{G}_\alpha, \mathbf{v} \rangle|^2 \\ &\quad - \left[2W + 2\Delta M \sum_{\alpha=1}^l (\|g''_\alpha\|_\infty^2 + \|g'_\alpha\|_\infty \|g''_\alpha\|_\infty) \right] \|\mathbf{v}\|^2 + 2s\theta' \left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 \right) \|\mathbf{v}\|^2 \\ &= \langle \mathbf{v}, \frac{\tilde{c}_1}{2} \mathcal{Q} \mathbf{v} \rangle + \langle \mathbf{v}, M \sum_{\alpha=1}^l |\mathbf{G}_\alpha\rangle \langle \mathbf{G}_\alpha| \mathbf{v} \rangle - \langle \mathbf{v}, (2W + 1) \mathbf{v} \rangle \\ &\quad + \left[1 - 2\Delta M \sum_{\alpha=1}^l (\|g''_\alpha\|_\infty^2 + \|g'_\alpha\|_\infty \|g''_\alpha\|_\infty) \right] \|\mathbf{v}\|^2 + 2s\theta' \left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 \right) \|\mathbf{v}\|^2 \\ &\geq \langle \mathbf{v}, \left(\frac{\tilde{c}_1}{2} \mathcal{Q} + M \sum_{\alpha=1}^l |\mathbf{G}_\alpha\rangle \langle \mathbf{G}_\alpha| - (2W + 1) \right) \mathbf{v} \rangle + (1 - C(M, l)\Delta) \|\mathbf{v}\|^2 \\ &\quad + \|\mathbf{v}\|^2 4s \left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 - 1 \right) \mathbf{1}_{\left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 > 1\right)} \end{aligned}$$

where $C(M, l) = 2M \sum_{\alpha=1}^l (\|g''_\alpha\|_\infty^2 + \|g'_\alpha\|_\infty \|g''_\alpha\|_\infty)$.

Now we determine the parameters.

1. Let $M = C_{**} + 2W + 1$.
2. By Lemma 12.1 there exist $0 < \epsilon < 0.1$, $l > 0$ and $N_0 > 0$ all depending on M and \tilde{c}_1 such that when $N > N_0$

$$\mathcal{Q} + \frac{2M}{\tilde{c}_1} \sum_{\alpha=1}^l |\mathbf{G}_\alpha\rangle \langle \mathbf{G}_\alpha| \geq \frac{2M}{\tilde{c}_1} \quad \text{and} \quad \mathcal{Q} + \frac{4M}{\tilde{c}_1} \sum_{\alpha=1}^l |\mathbf{G}_\alpha\rangle \langle \mathbf{G}_\alpha| \geq \frac{4M}{\tilde{c}_1}.$$

3. According to Lemma 12.3, there exists $s > 0$ depending on M and l such that

$$(1 - C(M, l) \max(x, \sqrt{x})) + 4s(sx - 1) \mathbf{1}_{(sx > 1)} > 0.$$

Since $\Delta \leq \max \left(\frac{1}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2, \sqrt{\frac{1}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2} \right)$, we have

$$(1 - C(M, l)\Delta) + 4s \left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 - 1 \right) \mathbb{1}_{\left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 > 1 \right)} > 0, \quad \forall \lambda_1, \dots, \lambda_N, \tilde{\gamma}_1, \dots, \tilde{\gamma}_N.$$

With the M, ϵ, l, s, N_0 chosen as above, for any $N > N_0$ and $\mathbf{v} \in \mathbb{R}^N$,

$$\langle \mathbf{v}, (\nabla^2 \mathcal{H}_\nu) \mathbf{v} \rangle \geq C_{**} \|\mathbf{v}\|^2.$$

Moreover M, ϵ, l, s, N_0 all depend only on C_{**} and \tilde{c}_1 . □

13 Log Sobolev inequality and concentration of ν_s and $\tilde{\mu}_s$

Suppose $\nu = \nu_N$ is a probability measure on $[a, b]^N$ with density $\frac{1}{Z_\nu} \exp(-\beta N \mathcal{H}_\nu)$ and $\nu_s = \nu_s(N)$ is a probability measures on $\tilde{\Sigma}_N$ with density $\frac{1}{Z_{\nu_s}} \exp(-N \beta \mathcal{H}_\nu)$ respectively where Z_{ν_s} and Z_ν^s are normalization constants.

13.1 Log Sobolev inequality

Lemma 13.1 (Log Sobolev inequality). *Suppose Ω is $\mathbb{R}^N, [a, b]^N$ or $\tilde{\Sigma}_N$. Suppose $\bar{V} : \Omega \rightarrow \mathbb{R} \cup \infty$ satisfies that $\bar{V}(x) - \frac{1}{2C_{\bar{V}}} \|x\|^2$ is convex. Suppose $\bar{\mu}$ is a probability measure on Ω with density $\frac{1}{Z_{\bar{V}}} e^{-\bar{V}(x)}$ where $Z_{\bar{V}}$ is the normalization constant. If F is a differentiable function of Ω , then*

$$\int_{\Omega} F^2 \ln \left(\frac{F^2}{\int_{\Omega} F^2 d\bar{\mu}} \right) d\bar{\mu} \leq 2C_{\bar{V}} \int_{\Omega} \|\nabla F\|_2^2 d\bar{\mu}.$$

Lemma 13.1 is a generalization of the log Sobolev inequality. It can be proved in the same way as Lemma 4.4 of [14].

Lemma 13.2. *Suppose Ω is $\mathbb{R}^N, [a, b]^N$ or $\tilde{\Sigma}_N$. Suppose $\bar{\mu}$ is a probability measure on Ω and \bar{c} is a constant satisfying*

$$\int_{\Omega} F^2 \ln \left(\frac{F^2}{\int_{\Omega} F^2 d\bar{\mu}} \right) d\bar{\mu} \leq 2\bar{c} \int_{\Omega} \|\nabla F\|_2^2 d\bar{\mu}$$

for every smooth function F on Ω . If $f(x)$ is a Lipschitz function on Ω with Lipschitz constant C_f , then for any $w > 0$,

$$\mathbb{P}^{\bar{\mu}}(|f(x) - \mathbb{E}^{\bar{\mu}} f(x)| \geq w) \leq 2 \exp\left(-\frac{w^2}{2\bar{c}C_f^2}\right).$$

Lemma 13.2 is a generalization of the Herbst's lemma. Using the argument of Lemma 4.4 of [14] one can prove Lemma 13.2 in the same way as Lemma 2.3.3 of [1].

Lemma 13.3. Suppose Ω is \mathbb{R}^N , $[a, b]^N$ or $\tilde{\Sigma}_N$. Suppose $\bar{V} : \Omega \rightarrow \mathbb{R} \cup \infty$ satisfies that $\bar{V}(x) - \frac{1}{2C_{\bar{V}}} \|x\|^2$ is convex. Suppose $\bar{\mu}$ is a probability measure on Ω with density $\frac{1}{Z_{\bar{V}}} e^{-\bar{V}(x)}$ where $Z_{\bar{V}}$ is the normalization constant. Suppose $f(x)$ is a Lipschitz function on Ω with Lipschitz constant C_f . Then for any $w > 0$,

$$\mathbb{P}^{\bar{\mu}}(|f(x) - \mathbb{E}^{\bar{\mu}} f(x)| \geq w) \leq 2 \exp\left(-\frac{w^2}{2C_{\bar{V}}C_f^2}\right).$$

Proof. Lemma 13.3 is an immediate sequence of Lemma 13.1 and Lemma 13.2. \square

13.2 Concentration of ν_s and $\tilde{\mu}_s$

Suppose $\tilde{c}_1 = \tilde{c}_1(V_p, \kappa, \epsilon_0)$ is as in Proposition 11.1 and $C_{**} = 2$. According to Lemma 12.2 there exist M, l, s, ϵ, N_0 all depending on V_p, κ and ϵ_0 but independent of $\{c_N\}$ such that if $N > N_0$ and $\lambda \in [a, b]^N$, then

$$\langle \mathbf{v}, (\nabla^2 \mathcal{H}_\nu) \mathbf{v} \rangle \geq 2 \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbb{R}^N. \quad (13.57)$$

From now on we fix these $\tilde{c}_1, M, l, s, \epsilon$ which all depend on V_p, κ and ϵ_0 but independent of $\{c_N\}$. So Proposition 11.1 and (13.57) are true for $N > N_0(V_p, \kappa, \epsilon_0)$.

For $1 \leq k \leq N$, $x \mapsto x_k$ is a Lipschitz function on $[a, b]^N$ with Lipschitz constant 1. According to Lemma 13.3 and (13.57) we have the following lemma.

Lemma 13.4. *There exists $N_0 = N_0(V_p, \kappa, \epsilon_0)$ such that when $N > N_0$ and $1 \leq k \leq N$,*

$$\mathbb{P}^\nu(|x_k - \mathbb{E}^\nu x_k| \geq N^{-\frac{1}{2}+w}) \leq 2 \exp(-\beta N^{2w}) \quad \forall w > 0,$$

$$\mathbb{P}^{\nu_s}(|x_k - \mathbb{E}^{\nu_s} x_k| \geq N^{-\frac{1}{2}+w}) \leq 2 \exp(-\beta N^{2w}) \quad \forall w > 0.$$

Corollary 13.5. There exists $N_0 = N_0(V_p, \kappa, \epsilon_0)$ such that when $N > N_0$ and $1 \leq k \leq N$,

$$\mathbb{E}^{\nu_s}(|\lambda_k - \mathbb{E}^{\nu_s} \lambda_k|^2) \leq 2N^{-1+\epsilon_0}.$$

Proof. By Lemma 13.4, there exists $N_0 = N_0(V_p, \kappa, \epsilon_0)$ such that when $N > N_0$ and $1 \leq k \leq N$,

$$\begin{aligned} & \mathbb{E}^{\nu_s}(|\lambda_k - \mathbb{E}^{\nu_s} \lambda_k|^2) \\ &= \int_{\tilde{\Sigma}_N} |\lambda_k - \mathbb{E}^{\nu_s} \lambda_k|^2 \mathbb{1}_{(|\lambda_k - \mathbb{E}^{\nu_s} \lambda_k| > N^{-\frac{1}{2}+0.5\epsilon_0})} d\nu_s + \int_{\tilde{\Sigma}_N} |\lambda_k - \mathbb{E}^{\nu_s} \lambda_k|^2 \mathbb{1}_{(|\lambda_k - \mathbb{E}^{\nu_s} \lambda_k| \leq N^{-\frac{1}{2}+0.5\epsilon_0})} d\nu_s \\ &\leq (2|a| + 2|b|)^2 \mathbb{P}^{\nu_s}(|\lambda_k - \mathbb{E}^{\nu_s} \lambda_k| > N^{-\frac{1}{2}+0.5\epsilon_0}) + N^{-1+\epsilon_0} \\ &\leq (2|a| + 2|b|)^2 \cdot 2 \exp(-\beta N^{\epsilon_0}) + N^{-\frac{1}{2}+\epsilon_0} \\ &\leq 2N^{-\frac{1}{2}+\epsilon_0}. \end{aligned}$$

\square

Lemma 13.6. *For any $0 < u_1 < u_2$, there exist $N_0 > 0$ depending on $V_p, \kappa, \epsilon_0, u_1$ and u_2 such that when $N > N_0$,*

$$\mathbb{P}^{\tilde{\mu}_s}(\beta N(\mathcal{H}_\nu - \mathcal{H}_{\tilde{\mu}}) > N^{2\epsilon_0+2u_2}) \leq \exp(-N^{\epsilon_0+u_1}). \quad (13.58)$$

Proof.

$$\begin{aligned}
& \mathbb{P}^{\tilde{\mu}_s}(\beta N(\mathcal{H}_\nu - \mathcal{H}_{\tilde{\mu}}) > N^{2\epsilon_0+2u_2}) \\
&= \mathbb{P}^{\tilde{\mu}_s} \left(\beta N^2 \theta \left(\frac{s}{N} \sum_1^N (\lambda_i - \tilde{\gamma}_i)^2 \right) + \beta \sum_{i,j} \theta \left(\sqrt{\tilde{c}_1 N Q_{i,j}} (\lambda_i - \lambda_j) \right) + \beta M \sum_{\alpha=1}^l \left[\sum_{j=1}^N (g_\alpha(\lambda_j) - g_\alpha(\tilde{\gamma}_j)) \right]^2 > N^{2\epsilon_0+2u_2} \right) \\
&\leq I + II + III \tag{13.59}
\end{aligned}$$

where

$$\begin{aligned}
I &= \mathbb{P}^{\tilde{\mu}_s} \left(\theta \left(\frac{s}{N} \sum_1^N (\lambda_i - \tilde{\gamma}_i)^2 \right) > \frac{1}{3\beta} N^{-2+2\epsilon_0+2u_2} \right); \\
II &= \mathbb{P}^{\tilde{\mu}_s} \left(\sum_{i,j} \theta \left(\sqrt{\tilde{c}_1 N Q_{i,j}} (\lambda_i - \lambda_j) \right) > \frac{1}{3\beta} N^{2\epsilon_0+2u_2} \right); \\
III &= \mathbb{P}^{\tilde{\mu}_s} \left(\sum_{\alpha=1}^l \left[\sum_{j=1}^N (g_\alpha(\lambda_j) - g_\alpha(\tilde{\gamma}_j)) \right]^2 > \frac{1}{3\beta M} N^{2\epsilon_0+2u_2} \right).
\end{aligned}$$

Notice that

$$I \leq \mathbb{P}^{\tilde{\mu}_s} \left(\frac{s}{N} \sum_1^N (\lambda_i - \tilde{\gamma}_i)^2 > 1 \right) \leq \mathbb{P}^{\tilde{\mu}_s} \left(\exists k \in [1, N] \text{ st. } |\lambda_k - \tilde{\gamma}_k| > \frac{1}{\sqrt{s}} \right).$$

According to Corollary 11.4, there exist $N_0 > 0$, $c_3 > 0$ depending on V_p , κ , ϵ_0 (since s is determined by V_p , κ , ϵ_0) such that if $N > N_0$ then $I \leq \exp(-c_3 N)$.

Notice that

$$II \leq \sum_{i,j} \mathbb{P}^{\tilde{\mu}_s} \left(\sqrt{\tilde{c}_1 N Q_{i,j}} (\lambda_i - \lambda_j) > 1 \right) = \sum_{i,j} \mathbb{P}^{\tilde{\mu}_s} \left(\frac{1}{N} \frac{1}{|\lambda_i - \lambda_j|^2} < \tilde{c}_1 Q_{i,j} \right).$$

According to Proposition 11.1 there exist $\tilde{c}_2 > 0$, $N_0 > 0$ depending on V_p , κ , ϵ_0 (since ϵ is determined by V_p , κ , ϵ_0) such that if $N > N_0$, then $II \leq N^2 \exp(-\tilde{c}_2 N)$.

To estimate *III*, we notice that if ϕ is a differentiable function with $\|\phi\|_\infty + \|\phi'\|_\infty < +\infty$, then

$$\begin{aligned}
& \left| \frac{1}{N} \sum_{i=1}^N \phi(\tilde{\gamma}_i) - \int \phi(t) \tilde{\rho}(t) dt \right| \\
&= \left| \frac{1}{2N} \phi(\tilde{\gamma}_1) + \int_c^{\tilde{\gamma}_1} \tilde{\rho}(s) \phi(\tilde{\gamma}_1) ds + \int_{\tilde{\gamma}_1}^{\tilde{\gamma}_2} \tilde{\rho}(s) \phi(\tilde{\gamma}_2) ds + \cdots + \int_{\tilde{\gamma}_{N-1}}^{\tilde{\gamma}_N} \tilde{\rho}(s) \phi(\tilde{\gamma}_N) ds \right. \\
&\quad \left. - \int_c^{\tilde{\gamma}_1} \tilde{\rho}(s) \phi(s) ds - \int_{\tilde{\gamma}_1}^{\tilde{\gamma}_2} \tilde{\rho}(s) \phi(s) ds - \cdots - \int_{\tilde{\gamma}_{N-1}}^{\tilde{\gamma}_N} \tilde{\rho}(s) \phi(s) ds - \int_{\tilde{\gamma}_N}^d \tilde{\rho}(s) \phi(s) ds \right| \\
&\leq \frac{1}{2N} \|\phi\|_\infty + \int_c^{\tilde{\gamma}_1} \tilde{\rho}(s) |\phi(s) - \phi(\tilde{\gamma}_1)| ds + \int_{\tilde{\gamma}_1}^{\tilde{\gamma}_2} \tilde{\rho}(s) |\phi(s) - \phi(\tilde{\gamma}_2)| ds + \cdots + \int_{\tilde{\gamma}_{N-1}}^{\tilde{\gamma}_N} \tilde{\rho}(s) |\phi(s) - \phi(\tilde{\gamma}_N)| ds \\
&\quad + \int_{\tilde{\gamma}_N}^d \tilde{\rho}(s) \|\phi\|_\infty ds \\
&\leq \frac{1}{2N} \|\phi\|_\infty + \|\phi'\|_\infty \frac{1}{2N} (\tilde{\gamma}_1 - c) + \|\phi'\|_\infty \frac{1}{N} (\tilde{\gamma}_2 - \tilde{\gamma}_1) + \cdots + \|\phi'\|_\infty \frac{1}{N} (\tilde{\gamma}_N - \tilde{\gamma}_{N-1}) + \frac{1}{2N} \|\phi\|_\infty \\
&\leq \frac{1}{N} (\|\phi\|_\infty + \|\phi'\|_\infty (d - c)). \tag{13.60}
\end{aligned}$$

Now we are ready to estimate *III*.

$$\begin{aligned}
III &\leq \sum_{\alpha=1}^l \mathbb{P}^{\tilde{\mu}_s} \left(\left| \sum_{j=1}^N (g_\alpha(\lambda_j) - g_\alpha(\tilde{\gamma}_j)) \right| > \frac{1}{\sqrt{3l\beta M}} N^{\epsilon_0+u_2} \right) \\
&\leq \sum_{\alpha=1}^l \left[\mathbb{P}^{\tilde{\mu}_s} \left(\left| \sum_{i=1}^N g_\alpha(\tilde{\gamma}_i) - N \int g_\alpha(t) \tilde{\rho}(t) dt \right| > \frac{1}{2\sqrt{3l\beta M}} N^{\epsilon_0+u_2} \right) \right. \\
&\quad \left. + \mathbb{P}^{\tilde{\mu}_s} \left(\left| \sum_{i=1}^N g_\alpha(\lambda_i) - N \int g_\alpha(t) \tilde{\rho}(t) dt \right| > \frac{1}{2\sqrt{3l\beta M}} N^{\epsilon_0+u_2} \right) \right]
\end{aligned}$$

According to (13.60), there exists $N_0 > 0$ depending on ϵ_0, u_2, M, l (M and l are determined by V_p, κ and ϵ_0) such that when $N > N_0$,

$$\sum_{\alpha=1}^l \mathbb{P}^{\tilde{\mu}_s} \left(\left| \sum_{i=1}^N g_\alpha(\tilde{\gamma}_i) - N \int g_\alpha(t) \tilde{\rho}(t) dt \right| > \frac{1}{2\sqrt{3l\beta M}} N^{\epsilon_0+u_2} \right) = 0.$$

According to Corollary 9.2, there exist $c_5 > 0, N_0 > 0$ depending on $u_1, u_2, V_p, \kappa, \epsilon_0, M, l$ (M and l are determined by V_p, κ and ϵ_0) such that when $N > N_0$,

$$\sum_{\alpha=1}^l \mathbb{P}^{\tilde{\mu}_s} \left(\left| \sum_{i=1}^N g_\alpha(\lambda_i) - N \int g_\alpha(t) \tilde{\rho}(t) dt \right| > \frac{1}{2\sqrt{3l\beta M}} N^{\epsilon_0+u_2} \right) \leq \exp(-N^{\epsilon_0+u_1}).$$

Thus $III \leq \exp(-N^{\epsilon_0+u_1})$ when $N > N_0(u_1, u_2, V_p, \kappa, \epsilon_0)$.

The estimations of *I, II* and *III* together complete the proof. \square

The following lemma is an analogue of Lemma 3.6 of [7].

Lemma 13.7. Suppose $\{A_N\}$ is a sequence of events on $\tilde{\Sigma}_N$.

- Suppose $u_2 > 2\epsilon_0$ and $0 < u_1 < u_2$. If there exist $c_1 > 0$, such that

$$\mathbb{P}^{\tilde{\mu}_s}(A_N) \leq c_1 \exp(-N^{u_2}) \quad \forall N,$$

then there exist $c_2 > 0$ depending on $c_1, u_1, u_2, V_p, \kappa$ and ϵ_0 such that for $N \geq 1$,

$$\mathbb{P}^{\nu_s}(A_N) \leq c_2 \exp(-N^{u_1}).$$

- Suppose $u_2 > 2\epsilon_0$ and $0 < u_1 < \frac{1}{2}u_2$. If there exist $c_1 > 0$, such that

$$\mathbb{P}^{\nu_s}(A_N) \leq c_1 \exp(-N^{u_2}) \quad \forall N,$$

then there exist $c_2 > 0$ depending on $c_1, u_1, u_2, V_p, \kappa$ and ϵ_0 such that for $N \geq 1$,

$$\mathbb{P}^{\tilde{\mu}_s}(A_N) \leq c_2 \exp(-N^{u_1}).$$

Proof. To prove the first statement, we use Jensen's inequality and have

$$\ln \int_{\tilde{\Sigma}_N} e^{\beta N(\mathcal{H}_{\tilde{\mu}} - \mathcal{H}_{\nu})} \frac{e^{-\beta N \mathcal{H}_{\tilde{\mu}}}}{\int_{\tilde{\Sigma}_N} e^{-\beta N \mathcal{H}_{\tilde{\mu}}} d\lambda} d\lambda \geq \int_{\tilde{\Sigma}_N} \beta N(\mathcal{H}_{\tilde{\mu}} - \mathcal{H}_{\nu}) \frac{e^{-\beta N \mathcal{H}_{\tilde{\mu}}}}{\int_{\tilde{\Sigma}_N} e^{-\beta N \mathcal{H}_{\tilde{\mu}}} d\lambda} d\lambda.$$

Therefore $\ln Z_{\tilde{\mu}_s} \leq \ln Z_{\nu_s} + \mathbb{E}^{\tilde{\mu}_s}(\beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}}))$ and $\frac{Z_{\tilde{\mu}_s}}{Z_{\nu_s}} \leq \exp(\mathbb{E}^{\tilde{\mu}_s}(\beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}})))$. According to Lemma 13.6, there exist $N_0 > 0$ depending on u_2, V_p, κ and ϵ_0 such that when $N > N_0$,

$$\mathbb{P}^{\tilde{\mu}_s}(\beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}}) > N^{2\epsilon_0+0.8(u_2-2\epsilon_0)}) \leq \exp(-N^{\epsilon_0+0.3(u_2-2\epsilon_0)}).$$

It is easy to check that there is $e_1 > 0$, $N_0 > 0$ depending on V_p, κ and ϵ_0 such that if $N > N_0$ then $(\beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}}) \leq e_1 N^3)$ everywhere on $[a, b]^N$. So

$$\begin{aligned} \mathbb{E}^{\tilde{\mu}_s}(\beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}})) &= \int_{\beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}}) > N^{2\epsilon_0+0.8(u_2-2\epsilon_0)}} \beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}}) d\tilde{\mu}_s + \int_{\beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}}) \leq N^{2\epsilon_0+0.8(u_2-2\epsilon_0)}} \beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}}) d\tilde{\mu}_s \\ &\leq N^{2\epsilon_0+0.8(u_2-2\epsilon_0)} + e_1 N^3 \exp(-N^{\epsilon_0+0.3(u_2-2\epsilon_0)}) \\ &\leq 2N^{2\epsilon_0+0.8(u_2-2\epsilon_0)} \end{aligned}$$

for $N > N_0(V_p, \kappa, \epsilon_0, u_2)$. Therefore

$$\mathbb{P}^{\nu_s}(A_N) \leq \frac{Z_{\tilde{\mu}_s}}{Z_{\nu_s}} \mathbb{P}^{\tilde{\mu}_s}(A_N) \leq c_1 \exp(2N^{2\epsilon_0+0.8(u_2-2\epsilon_0)} - N^{u_2}) \leq c_1 \exp(-N^{u_1})$$

for $N > N_0(V_p, \kappa, \epsilon_0, u_1, u_2)$. We can choose c_2 appropriately to make the first statement true.

To prove the second statement, we notice that $\mathcal{H}_{\nu} \geq \mathcal{H}_{\tilde{\mu}}$, thus $Z_{\nu_s} \leq Z_{\tilde{\mu}_s}$ and

$$\begin{aligned} \mathbb{P}^{\tilde{\mu}_s}(A_N) &= \mathbb{P}^{\tilde{\mu}_s}(A_N \cap \{\beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}}) > N^{u_1 \vee \epsilon_0 + 0.5u_2}\}) + \mathbb{P}^{\tilde{\mu}_s}(A_N \cap \{\beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}}) \leq N^{u_1 \vee \epsilon_0 + 0.5u_2}\}) \\ &\leq \mathbb{P}^{\tilde{\mu}_s}(\beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}}) > N^{u_1 \vee \epsilon_0 + 0.5u_2}) + \exp(N^{u_1 \vee \epsilon_0 + 0.5u_2}) \mathbb{P}^{\nu}(A_N) \\ &\leq \mathbb{P}^{\tilde{\mu}_s}(\beta N(\mathcal{H}_{\nu} - \mathcal{H}_{\tilde{\mu}}) > N^{u_1 \vee \epsilon_0 + 0.5u_2}) + c_1 \exp(N^{u_1 \vee \epsilon_0 + 0.5u_2} - N^{u_2}) \end{aligned} \tag{13.61}$$

where $u_1 \vee \epsilon_0 := \max(u_1, \epsilon_0)$. From (13.61) and Lemma 13.6, there must be $c_2 > 0$ depending on $c_1, u_1, u_2, V_p, \kappa$ and ϵ_0 such that for $N \geq 1$,

$$\mathbb{P}^{\tilde{\mu}_s}(A_N) \leq c_2 \exp(-N^{u_1}).$$

□

Suppose $w_1 < w_2$ and $w_2 > \epsilon_0$. Set $A_N = \{x \in \tilde{\Sigma}_N | \exists k \in [1, N] \text{ st. } |x_k - \mathbb{E}^{\nu_s}(x_k)| \geq N^{-\frac{1}{2}+w_2}\}$. From Lemma 13.4 and Lemma 13.7, there exist $c_3 > 0$ depending on w_1, w_2, V_p, κ and ϵ_0 such that when $N \geq 1$,

$$\mathbb{P}^{\tilde{\mu}_s}(\exists k \in [1, N] \text{ st. } |x_k - \mathbb{E}^{\nu_s}(x_k)| \geq N^{-\frac{1}{2}+w_2}) = \mathbb{P}^{\tilde{\mu}_s}(A_N) \leq c_3 \exp(-N^{w_1}).$$

Notice that

$$\begin{aligned} |\mathbb{E}^{\tilde{\mu}_s}(x_k) - \mathbb{E}^{\nu_s}(x_k)| &= \left| \int_{\tilde{\Sigma}_N} x_k - \mathbb{E}^{\nu_s}(x_k) d\tilde{\mu}_s(x) \right| \leq \int_{A_N} |x_k - \mathbb{E}^{\nu_s}(x_k)| d\tilde{\mu}_s(x) + \int_{A_N^c} |x_k - \mathbb{E}^{\nu_s}(x_k)| d\tilde{\mu}_s(x) \\ &\leq 2 \max(|a|, |b|) \mathbb{P}^{\tilde{\mu}_s}(A_N) + N^{-\frac{1}{2}+w_2} \leq 2 \max(|a|, |b|) c_3 \exp(-N^{w_1}) + N^{-\frac{1}{2}+w_2}. \end{aligned}$$

Thus there exists $N_0 > 0$ depending on w_1, w_2, V_p, κ and ϵ_0 such that if $N > N_0$, then $|\mathbb{E}^{\tilde{\mu}_s}(x_k) - \mathbb{E}^{\nu_s}(x_k)| \leq 2N^{-\frac{1}{2}+w_2}$ and we have:

Lemma 13.8. *Suppose $w_1 < w_2$ and $w_2 > \epsilon_0$. There exist $c_3 > 0, N_0 > 0$ depending on w_1, w_2, V_p, κ and ϵ_0 such that if $N > N_0$, then*

$$\mathbb{P}^{\tilde{\mu}_s}(\exists k \in [1, N] \text{ st. } |x_k - \mathbb{E}^{\tilde{\mu}_s}(x_k)| \geq 3N^{-\frac{1}{2}+w_2}) \leq c_3 \exp(-N^{w_1}).$$

Corollary 13.9. *There exists $N_0 > 0$ depending on V_p, κ and ϵ_0 such that if $N > N_0$ and $k \in [1, N]$, then*

$$\mathbb{E}^{\tilde{\mu}_s}(|x_k - \mathbb{E}^{\tilde{\mu}_s} x_k|^2) \leq N^{-1+2.3\epsilon_0}.$$

Proof. Set

$$A = \{\lambda \in \tilde{\Sigma}_N | \exists k \in [1, N] \text{ st. } |x_k - \mathbb{E}^{\tilde{\mu}_s} x_k| \geq 3N^{-0.5+1.1\epsilon_0}\}.$$

According to Lemma 13.8, $\mathbb{P}^{\tilde{\mu}_s}(A) \leq \exp(-N^{\epsilon_0})$ when $N > N_0(V_p, \kappa, \epsilon_0)$.

$$\begin{aligned} \mathbb{E}^{\tilde{\mu}_s}(|x_k - \mathbb{E}^{\tilde{\mu}_s} x_k|^2) &= \int_A |x_k - \mathbb{E}^{\tilde{\mu}_s} x_k|^2 d\tilde{\mu}_s + \int_{A^c} |x_k - \mathbb{E}^{\tilde{\mu}_s} x_k|^2 d\tilde{\mu}_s \\ &\leq (2|a| + 2|b|)^2 \exp(-N^{\epsilon_0}) + 9N^{-1+2.2\epsilon_0} \\ &\leq N^{-1+2.3\epsilon_0}. \end{aligned}$$

□

14 Estimation of $|\gamma_k - \gamma_k^{(N)}|$ of scale $N^{-\frac{1}{2}}$

Recall that γ_k is defined by

$$\int_c^{\gamma_k} \bar{\rho}(t) dt = \frac{k}{N}.$$

Similarly we defined $\gamma_k^{(N)}$ by

$$\int_c^{\gamma_k^{(N)}} \tilde{\rho}_1^{(N)}(t) dt = \frac{k}{N}$$

where $\tilde{\rho}_1^{(N)}(t)$ is the one point correlation function of $\tilde{\mu}$.

We need the following technical lemma about Helffer-Sjöstrand functional calculus. It is induced from Lemma 9.9 with $D_\chi = 10\kappa$.

Lemma 14.1. *Suppose $\chi(x) : \mathbb{R} \rightarrow [-1, 1]$ is a smooth even function with $\chi(x) = 1$ on $[-5\kappa, 5\kappa]$ and $\chi(x) = 0$ on $[-10\kappa, 10\kappa]$. Suppose $f(x)$ is a C^2 function with compact support. Suppose $\bar{\rho}(t)dt$ is a signed measure with Stieltjes transform $\bar{m}(x + iy) = \int \frac{1}{x + iy - t} \bar{\rho}(t) dt$. Suppose $\int |\bar{\rho}(t)| dt < +\infty$. We have*

$$\begin{aligned} \left| \int f(t) \bar{\rho}(t) dt \right| &\leq \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} y f''(x) \chi(y) \operatorname{Im} \bar{m}(x + iy) dx dy \right| \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{R}^2} (|f(x)| + |y| |f'(x)|) |\chi'(y)| |\bar{m}(x + iy)| dx dy \end{aligned}$$

Suppose $\psi(x)$ is a smooth function satisfying the following conditions.

1. $\psi(x) = 1$ if $x \leq 0$.
2. $\psi(x) = 0$ if $x \geq 1$
3. $\psi(x) \in [0, 1]$ for all $x \in \mathbb{R}$.

Set $\eta_0 = \eta_0(N) = N^{-\frac{1}{2} + \epsilon_0}$. For $E \in [c - \frac{\kappa}{4}, d + \frac{\kappa}{4}]$, set

$$f_E(x) = \begin{cases} \psi(\frac{x-E}{2\eta_0} + \frac{1}{2}) & \text{if } x \geq a \\ 1 & \text{if } a - \sqrt{N} \leq x \leq a \\ \psi(a - \sqrt{N} - x) & \text{if } x \leq a - \sqrt{N} \end{cases}$$

So

$$f_E(x) = \begin{cases} 0 & \text{if } x \in (-\infty, a - \sqrt{N} - 1] \cup [E + \eta_0, +\infty) \\ 1 & \text{if } x \in [a - \sqrt{N}, E - \eta_0]. \end{cases}$$

Moreover, if $a - \sqrt{N} - 1 \leq x \leq a - \sqrt{N}$, then $|f'_E(x)| \leq \|\psi'\|_\infty$ and $|f''_E(x)| \leq \|\psi''\|_\infty$; if $E - \eta_0 \leq x \leq E + \eta_0$, then $|f'_E(x)| \leq \frac{1}{2\eta_0} \|\psi'\|_\infty$ and $|f''_E(x)| \leq \frac{1}{4\eta_0^2} \|\psi''\|_\infty$. Suppose $\bar{\rho}(t) = \tilde{\rho}(t) - \tilde{\rho}_1^{(N)}(t)$ and $\bar{m}(z) = \int_a^b \frac{1}{z-t} \bar{\rho}(t) dt$.

Lemma 14.2. *There exist $T_1 = T_1(V_p, \kappa, \epsilon_0) > 0$ and $N_0 = N_0(V_p, \kappa, \epsilon_0) > 0$ such that if $N > N_0$ and $E \in [c - \frac{\kappa}{4}, d + \frac{\kappa}{4}]$ (E may depend on N), then*

$$|\int f_E(t)\bar{\rho}(t)dt| \leq T_1 \cdot N^{-\frac{1}{2}+2\epsilon_0}.$$

Proof. By Lemma 14.1 we have:

$$\begin{aligned} & |\int f_E(t)\bar{\rho}(t)dt| \\ & \leq \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} y f_E''(x) \chi(y) \text{Im} \bar{m}(x + iy) dx dy \right| \\ & + \frac{1}{2\pi} \int_{\mathbb{R}^2} |f_E(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy + \frac{1}{2\pi} \int_{\mathbb{R}^2} |y| |f_E'(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy \\ & = \frac{1}{\pi} \left| \int_0^{10\kappa} \int_{a-\sqrt{N}-1}^{a-\sqrt{N}} y f_E''(x) \chi(y) \text{Im} \bar{m}(x + iy) dx dy \right| + \frac{1}{\pi} \left| \int_0^{10\kappa} \int_{E-\eta_0}^{E+\eta_0} y f_E''(x) \chi(y) \text{Im} \bar{m}(x + iy) dx dy \right| \\ & + \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{a-\sqrt{N}-1}^{E+\eta_0} |f_E(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy + \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{a-\sqrt{N}-1}^{a-\sqrt{N}} |y| |f_E'(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy \\ & + \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{E-\eta_0}^{E+\eta_0} |y| |f_E'(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy \\ & := I + II + III + IV + V. \end{aligned}$$

• **Estimation of I and IV**

When $x \in [a - \sqrt{N} - 1, a - \sqrt{N}]$, the distance between x and $[a, b]$ is no less than \sqrt{N} , so $|\bar{m}(x + iy)| \leq N^{-1/2}$ for $\forall y$. So we have:

$$I \leq \frac{100\kappa^2}{\pi} \|\psi''\|_{\infty} N^{-1/2} \quad \text{and} \quad IV \leq \frac{50\kappa^2}{\pi} \|\psi'\|_{\infty} \|\chi'\|_{\infty} N^{-1/2}.$$

• **Estimation of III and V**

Fix $k_0 = k_0(\epsilon_0) \in \mathbb{N}$ such that $(\frac{1}{2})^{k_0} < \frac{1}{4}\epsilon_0$. According to Theorem 9.6 with $h \equiv 0$, there exist $C_{ind} = C_{ind}(V_p, \kappa, \epsilon_0) > 0$ and $N_0 = N_0(V_p, \kappa, \epsilon_0) > 0$ such that for any $N > N_0$ and $\text{Im} z \in [N^{-a_{k_0}}, 10\kappa)$,

$$|(z - c)(z - d)|^{1/2} |\tilde{m}_N(z) - \tilde{m}(z)| \leq N^{-1+\frac{3}{2}\epsilon_0} \ln N \cdot C_{ind}.$$

If $N > N_1(\kappa, \epsilon_0)$, then $\ln N < N^{\epsilon_0/2}$ and $[5\kappa, 10\kappa) \subset [N^{-a_{k_0}}, 10\kappa)$ thus $|(z - c)(z - d)|^{1/2} \geq 5\kappa$ for $z \in \{z | \text{Im} z \geq 5\kappa\}$. Therefore when $N > N_0(V_p, \kappa, \epsilon_0)$ we have:

$$\begin{aligned}
III &= \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{a-\sqrt{N}-1}^{E+\eta_0} |f_E(x)| |\chi'(y)| |\bar{m}(x+iy)| dx dy \\
&\leq \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{a-\sqrt{N}-1}^{E+\eta_0} |f_E(x)| |\chi'(y)| \left(N^{-1+2\epsilon_0} \frac{1}{5\kappa} C_{ind} \right) dx dy \\
&\leq \frac{1}{\pi} C_{ind} \|\chi'\|_{\infty} (d-c+\kappa+2+\sqrt{N}) N^{-1+2\epsilon_0} \\
&\leq \frac{2}{\pi} C_{ind} \|\chi'\|_{\infty} N^{-\frac{1}{2}+2\epsilon_0},
\end{aligned}$$

$$\begin{aligned}
V &= \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{E-\eta_0}^{E+\eta_0} |y| |f'_E(x)| |\chi'(y)| |\bar{m}(x+iy)| dx dy \\
&\leq \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{E-\eta_0}^{E+\eta_0} |y| |f'_E(x)| |\chi'(y)| \left(N^{-1+2\epsilon_0} \frac{1}{5\kappa} C_{ind} \right) dx dy \\
&\leq \frac{10\kappa}{\pi} C_{ind} \|\psi'\|_{\infty} \|\chi'\|_{\infty} N^{-1+2\epsilon_0}.
\end{aligned}$$

• **Estimation of II**

To estimate II , notice that $II \leq VI + VII$ where

$$\begin{aligned}
VI &= \frac{1}{\pi} \left| \int_0^{N^{-\frac{1}{2}+\epsilon_0}} \int_{E-\eta_0}^{E+\eta_0} y f''_E(x) \chi(y) \text{Im} \bar{m}(x+iy) dx dy \right|, \\
VII &= \frac{1}{\pi} \left| \int_{N^{-\frac{1}{2}+\epsilon_0}}^{10\kappa} \int_{E-\eta_0}^{E+\eta_0} y f''_E(x) \chi(y) \text{Im} \bar{m}(x+iy) dx dy \right|.
\end{aligned}$$

• **Estimation of VI**

Set $y_0 = N^{-\frac{1}{2}+\epsilon_0}$. Suppose $x \in [E-\eta_0, E+\eta_0]$ and $y \in (0, y_0]$. By direct computation, $y |\text{Im} \tilde{m}_N(x+iy)| \leq y_0 |\text{Im} \tilde{m}_N(x+iy_0)|$. Then,

$$\begin{aligned}
y |\text{Im} \bar{m}(x+iy)| &= y |\text{Im}(\tilde{m}_N(x+iy) - \tilde{m}(x+iy))| \leq y |\text{Im}(\tilde{m}_N(x+iy))| + y \pi \|\tilde{\rho}\|_{\infty} \quad (\text{see Lemma 10.2}). \\
&\leq y_0 |\text{Im}(\tilde{m}_N(x+iy_0))| + y_0 \pi \|\tilde{\rho}\|_{\infty} \leq y_0 |\text{Im}(\tilde{m}_N(x+iy_0) - \tilde{m}(x+iy_0))| + 2y_0 \pi \|\tilde{\rho}\|_{\infty} \\
&\leq y_0 |\tilde{m}_N(x+iy_0) - \tilde{m}(x+iy_0)| + 2y_0 \pi \|\tilde{\rho}\|_{\infty}.
\end{aligned}$$

According to Lemma 9.4 with $h \equiv 0$, there exist $C > 0$, $N_0 > 0$ depending on V_p , ϵ_0 and κ such that if $N > N_0$ and $x \in [E-\eta_0, E+\eta_0]$, then $|\tilde{m}_N(x+iy_0) - \tilde{m}(x+iy_0)| \leq \frac{C}{y_0} \sqrt{\frac{\ln N}{N}}$ and therefore for $y \in (0, y_0]$,

$$y |\text{Im} \bar{m}(x+iy)| \leq y_0 \frac{C}{y_0} \sqrt{\frac{\ln N}{N}} + 2y_0 \pi \|\tilde{\rho}\|_{\infty} = C \sqrt{\frac{\ln N}{N}} + 2\pi \|\tilde{\rho}\|_{\infty} N^{-\frac{1}{2}+\epsilon_0} \leq 3\pi \|\tilde{\rho}\|_{\infty} N^{-\frac{1}{2}+\epsilon_0}.$$

Therefore for $N > N_0(V_p, \kappa, \epsilon_0)$,

$$\begin{aligned} VI &\leq \frac{1}{\pi} \left| \int_0^{N^{-\frac{1}{2}+\epsilon_0}} \int_{E-\eta_0}^{E+\eta_0} f_E''(x) \chi(y) \left(3\pi \|\tilde{\rho}\|_\infty N^{-\frac{1}{2}+\epsilon_0} \right) dx dy \right| = \frac{3}{2} \|\tilde{\rho}\|_\infty \|\psi''\|_\infty N^{-1+2\epsilon_0} \frac{1}{\eta_0} \\ &= \frac{3}{2} \|\tilde{\rho}\|_\infty \|\psi''\|_\infty N^{-\frac{1}{2}+\epsilon_0} \quad (\text{since } \eta_0 = N^{-\frac{1}{2}+\epsilon_0}). \end{aligned}$$

• **Estimation of VII**

To estimate VII, notice that $\bar{m}(z)$ is analytic on a neighborhood of the domain of the integral: $\{x + iy | x \in (E - \eta_0, E + \eta_0), y \in (N^{-\frac{1}{2}+\epsilon_0}, 10\kappa)\}$. So on this domain $\frac{\partial}{\partial x} \text{Im} \bar{m}(x + iy) = -\frac{\partial}{\partial y} \text{Re} \bar{m}(x + iy)$. For $N > N_0(\kappa, \epsilon_0)$, we have $N^{-\frac{1}{2}+\epsilon_0} < 5\kappa$ and

$$\begin{aligned} VII &= \frac{1}{\pi} \left| \int_{N^{-\frac{1}{2}+\epsilon_0}}^{10\kappa} \int_{E-\eta_0}^{E+\eta_0} y f_E'(x) \chi(y) \frac{\partial}{\partial x} (\text{Im} \bar{m}(x + iy)) dx dy \right| \quad (\text{integral by parts}) \\ &= \frac{1}{\pi} \left| \int_{N^{-\frac{1}{2}+\epsilon_0}}^{10\kappa} \int_{E-\eta_0}^{E+\eta_0} y f_E'(x) \chi(y) \frac{\partial}{\partial y} (\text{Re} \bar{m}(x + iy)) dx dy \right| \\ &\leq \frac{1}{\pi} \left| \int_{E-\eta_0}^{E+\eta_0} \left(\text{Re} \bar{m}(x + iN^{-\frac{1}{2}+\epsilon_0}) \right) N^{-\frac{1}{2}+\epsilon_0} f_E'(x) dx \right| \\ &\quad + \frac{1}{\pi} \left| \int_{E-\eta_0}^{E+\eta_0} \int_{N^{-\frac{1}{2}+\epsilon_0}}^{10\kappa} \left(\text{Re} \bar{m}(x + iy) \right) \frac{\partial}{\partial y} (y \chi(y)) f_E'(x) dy dx \right|. \end{aligned}$$

By Lemma 9.4 with $h \equiv 0$, there are $C > 0$, $N_0 > 0$ depending on V_p , κ and ϵ_0 such that if $N > N_0$ and $y > 0$ then $|\bar{m}(x + iy)| \leq \frac{C}{y} \sqrt{\frac{\ln N}{N}}$ and

$$\begin{aligned} VII &\leq \frac{1}{\pi} \left| \int_{E-\eta_0}^{E+\eta_0} \frac{C}{N^{-\frac{1}{2}+\epsilon_0}} \sqrt{\frac{\ln N}{N}} N^{-\frac{1}{2}+\epsilon_0} f_E'(x) dx \right| + \frac{C}{\pi} \sqrt{\frac{\ln N}{N}} \left| \int_{E-\eta_0}^{E+\eta_0} \int_{N^{-\frac{1}{2}+\epsilon_0}}^{10\kappa} \frac{1}{y} \frac{\partial}{\partial y} (y \chi(y)) f_E'(x) dy dx \right| \\ &\leq \frac{C}{\pi} \sqrt{\frac{\ln N}{N}} \|\psi'\|_\infty + \frac{C}{\pi} \sqrt{\frac{\ln N}{N}} \|\psi'\|_\infty (1 + 10\kappa \|\chi'\|_\infty) \int_{N^{-\frac{1}{2}+\epsilon_0}}^{10\kappa} \frac{1}{y} dy \\ &= \frac{C}{\pi} \sqrt{\frac{\ln N}{N}} \|\psi'\|_\infty + \frac{C}{\pi} \sqrt{\frac{\ln N}{N}} \|\psi'\|_\infty (1 + 10\kappa \|\chi'\|_\infty) (\ln(10\kappa) - (-\frac{1}{2} + \epsilon_0) \ln N) \\ &\leq M_0 (\ln N)^{3/2} N^{-1/2} \end{aligned}$$

where M_0 depends on V_p , κ and ϵ_0 .

The estimations of I-VII complete the proof. \square

Lemma 14.3. *For any $\alpha \in (0, 1/2)$, there exist $N_0 = N_0(V_p, \kappa, \epsilon_0, \alpha) > 0$ and $T_2 = T_2(V_p, \kappa, \epsilon_0, \alpha) > 0$ such that if $N > N_0$ and $k \in [\alpha N, (1 - \alpha)N]$, then $|\gamma_k^{(N)} - \gamma_k| \leq T_2 N^{-\frac{1}{2}+2\epsilon_0}$.*

Proof. According to Lemma 14.2, there exist $T_1 = T_1(V_p, \kappa, \epsilon_0) > 0$ and $N_0 = N_0(V_p, \kappa, \epsilon_0) > 0$ such that if $N > N_0$ and $E \in [c, d]$, then $E \pm \eta_0 \in [c - \kappa/4, d + \kappa/4]$ and

$$\left| \int f_{E \pm \eta_0}(t) \bar{\rho}(t) dt \right| \leq T_1 \cdot N^{-\frac{1}{2}+2\epsilon_0}.$$

Thus when $N > N_0$ and $E \in [c, d]$,

$$\begin{aligned} \int_{-\infty}^E \tilde{\rho}_1^{(N)}(t) dt &\leq \int_{\mathbb{R}} \tilde{\rho}_1^{(N)}(t) f_{E+\eta_0}(t) dt = \int_{\mathbb{R}} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) f_{E+\eta_0}(t) dt + \int_{\mathbb{R}} \tilde{\rho}(t) f_{E+\eta_0}(t) dt \\ &\leq T_1 N^{-\frac{1}{2}+2\epsilon_0} + \int_{\mathbb{R}} \tilde{\rho}(t) f_{E+\eta_0}(t) dt = T_1 N^{-\frac{1}{2}+2\epsilon_0} + \int_{-\infty}^E \tilde{\rho}(t) dt + \int_E^{E+2\eta_0} \tilde{\rho}(t) f_{E+\eta_0}(t) dt \\ &\leq \int_{-\infty}^E \tilde{\rho}(t) dt + T_1 N^{-\frac{1}{2}+2\epsilon_0} + 2\|\tilde{\rho}\|_{\infty} N^{-\frac{1}{2}+\epsilon_0} \leq \int_{-\infty}^E \tilde{\rho}(t) dt + 2T_1 N^{-\frac{1}{2}+2\epsilon_0}, \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^E \tilde{\rho}_1^{(N)}(t) dt &\geq \int_{\mathbb{R}} \tilde{\rho}_1^{(N)}(t) f_{E-\eta_0}(t) dt = \int_{\mathbb{R}} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) f_{E-\eta_0}(t) dt + \int_{\mathbb{R}} \tilde{\rho}(t) f_{E-\eta_0}(t) dt \\ &\geq \int_{\mathbb{R}} \tilde{\rho}(t) f_{E-\eta_0}(t) dt - T_1 N^{-\frac{1}{2}+2\epsilon_0} = \int_{-\infty}^E \tilde{\rho}(t) dt - \int_{E-2\eta_0}^E \tilde{\rho}(t) (1 - f_{E-\eta_0}(t)) dt - T_1 N^{-\frac{1}{2}+2\epsilon_0} \\ &\geq \int_{-\infty}^E \tilde{\rho}(t) dt - T_1 N^{-\frac{1}{2}+2\epsilon_0} - 2\|\tilde{\rho}\|_{\infty} N^{-\frac{1}{2}+\epsilon_0} \geq \int_{-\infty}^E \tilde{\rho}(t) dt - 2T_1 N^{-\frac{1}{2}+2\epsilon_0}, \end{aligned}$$

and thus

$$\left| \int_{-\infty}^E \tilde{\rho}_1^{(N)}(t) dt - \int_{-\infty}^E \tilde{\rho}(t) dt \right| \leq 2T_1 N^{-\frac{1}{2}+2\epsilon_0}. \quad (14.62)$$

For any $\alpha \in (0, 1/2)$, there exists $s = s(V_p, \alpha) > 0$ such that if $k \in [\frac{\alpha}{2}N, (1 - \frac{\alpha}{2})N]$, then $\gamma_k \in [c + s, d - s]$. If $k \in [\alpha N, (1 - \alpha)N]$, then according to Lemma 14.2, for every $N > N_0(V_p, \kappa, \epsilon_0, \alpha)$,

$$\begin{aligned} \int_{-\infty}^{\gamma_k + 0.5\alpha N} \tilde{\rho}_1^{(N)}(t) dt &= \int_{-\infty}^{\gamma_k + 0.5\alpha N} \tilde{\rho}(t) dt + \int_{-\infty}^{\gamma_k + 0.5\alpha N} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) dt \geq \frac{k}{N} + 0.5\alpha - T_1 N^{-\frac{1}{2}+2\epsilon_0} \geq \frac{k}{N}, \\ \int_{-\infty}^{\gamma_k - 0.5\alpha N} \tilde{\rho}_1^{(N)}(t) dt &= \int_{-\infty}^{\gamma_k - 0.5\alpha N} \tilde{\rho}(t) dt + \int_{-\infty}^{\gamma_k - 0.5\alpha N} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) dt \leq \frac{k}{N} - 0.5\alpha + T_1 N^{-\frac{1}{2}+2\epsilon_0} \leq \frac{k}{N} \end{aligned}$$

and thus $\gamma_k^{(N)} \in [\gamma_k - 0.5\alpha N, \gamma_k + 0.5\alpha N] \subset [c + s, d - s]$.

So for $N > N_0(V_p, \kappa, \epsilon_0, \alpha)$ and $k \in [\alpha N, (1 - \alpha)N]$,

$$\left| \int_{-\infty}^{\gamma_k^{(N)}} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) dt \right| = \left| \int_{\gamma_k^{(N)}}^{\gamma_k} \tilde{\rho}(t) dt \right| \geq |\gamma_k^{(N)} - \gamma_k| \min_{[c+s, d-s]} \tilde{\rho}(t)$$

and by (14.62) we have

$$|\gamma_k^{(N)} - \gamma_k| \leq \left(\min_{[c+s, d-s]} \tilde{\rho}(t) \right)^{-1} \left| \int_{-\infty}^{\gamma_k^{(N)}} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) dt \right| \leq 2T_1 N^{-\frac{1}{2}+2\epsilon_0} \left(\min_{[c+s, d-s]} \tilde{\rho}(t) \right)^{-1}.$$

Setting $T_2 = 2T_1 \left(\min_{[c+s, d-s]} \tilde{\rho}(t) \right)^{-1}$ we complete the proof. \square

15 Rigidity at scale $N^{-1/2}$

Theorem 15.1. *Suppose $w_1 < w_2$ and $w_2 > 2\epsilon_0$. For any $\alpha \in (0, 1/2)$, there exist $N_0 = N_0(V_p, \kappa, \epsilon_0, \alpha, w_1, w_2) > 0$ such that for every $N > N_0$,*

$$\mathbb{P}^{\tilde{\mu}_s} \left(\exists k \in [\alpha N, (1 - \alpha)N] \text{ st. } |\lambda_k - \gamma_k| > N^{-\frac{1}{2} + w_2} \right) \leq \exp(-N^{w_1}).$$

Proof. Fix $\alpha \in (0, 1/2)$. Suppose $w_1 < w_2$ and $w_2 > 2\epsilon_0$. According to Lemma 13.8, there exists $N_0 > 0$ depending on w_1, w_2, V_p, κ and ϵ_0 such that if $N > N_0$, then

$$\mathbb{P}^{\tilde{\mu}_s} \left(\exists k \in [1, N] \text{ st. } |x_k - \mathbb{E}^{\tilde{\mu}_s}(x_k)| \geq N^{-\frac{1}{2} + w_2} \right) \leq \exp(-N^{w_1}). \quad (15.63)$$

Now we use a similar argument as the proof of Theorem 2.4 of [8].

Suppose $N > N_0$ and $k_1 \in [1, N]$. Notice that $\lambda \mapsto \#\{i | \lambda_i \leq \gamma_{k_1}^{(N)}\}$ is a symmetric function on $[a, b]^N$. So by Lemma 5.2,

$$\begin{aligned} \mathbb{E}^{\tilde{\mu}_s} \left[\#\{i | \lambda_i \leq \gamma_{k_1}^{(N)}\} \right] &= \mathbb{E}^{\tilde{\mu}} \left[\#\{i | \lambda_i \leq \gamma_{k_1}^{(N)}\} \right] = \mathbb{E}^{\tilde{\mu}} \left[\sum_{i=1}^N \mathbf{1}_{(-\infty, \gamma_{k_1}^{(N)})}(\lambda_i) \right] \\ &= N \int_{-\infty}^{\gamma_{k_1}^{(N)}} \tilde{\rho}_1^{(N)}(t) dt \\ &= k_1. \end{aligned} \quad (15.64)$$

On the other hand, according to (15.63), when $N > N_0(w_1, w_2, V_p, \kappa, \epsilon_0)$,

$$\begin{aligned} \mathbb{E}^{\tilde{\mu}_s} \left[\#\{i | \lambda_i \leq \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) - N^{-\frac{1}{2} + w_2}\} \right] &= \sum_{l=1}^N \mathbb{P}^{\tilde{\mu}_s}(\lambda_l \leq \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) - N^{-\frac{1}{2} + w_2}) \\ &\leq k_1 - 1 + (N - k_1 + 1) \mathbb{P}^{\tilde{\mu}_s}(\lambda_{k_1} \leq \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) - N^{-\frac{1}{2} + w_2}) \leq k_1 - 1 + (N - k_1 + 1) \exp(-N^{w_1}) \\ &\leq k_1 \end{aligned} \quad (15.65)$$

Similarly,

$$\begin{aligned} \mathbb{E}^{\tilde{\mu}_s} \left[\#\{i | \lambda_i > \gamma_{k_1-1}^{(N)} - \frac{1}{2} N^{-\frac{1}{2} + w_2}\} \right] &= N - \mathbb{E}^{\tilde{\mu}} \left[\#\{i | \lambda_i \leq \gamma_{k_1-1}^{(N)} - \frac{1}{2} N^{-\frac{1}{2} + w_2}\} \right] \\ &= N - \mathbb{E}^{\tilde{\mu}} \left[\sum_{i=1}^N \mathbf{1}_{(-\infty, \gamma_{k_1-1}^{(N)} - \frac{1}{2} N^{-\frac{1}{2} + w_2})}(\lambda_i) \right] \\ &\geq N - N \int_{-\infty}^{\gamma_{k_1-1}^{(N)}} \tilde{\rho}_1^{(N)}(t) dt \\ &= N - (k_1 - 1). \end{aligned} \quad (15.66)$$

and when $N > N_0(w_1, w_2, V_p, \kappa, \epsilon_0)$,

$$\begin{aligned} \mathbb{E}^{\tilde{\mu}_s} \left[\#\{i | \lambda_i > \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) + N^{-\frac{1}{2} + w_2}\} \right] &= \sum_{l=1}^N \mathbb{P}^{\tilde{\mu}_s}(\lambda_l > \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) + N^{-\frac{1}{2} + w_2}) \\ &\leq N - k_1 + k_1 \mathbb{P}^{\tilde{\mu}_s}(\lambda_{k_1} > \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) + N^{-\frac{1}{2} + w_2}) \leq N - k_1 + k_1 \exp(-N^{w_1}) \\ &\leq N - k_1 + 1. \end{aligned} \quad (15.67)$$

By (15.64) and (15.65),

$$\gamma_{k_1}^{(N)} \geq \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) - N^{-\frac{1}{2}+w_2}. \quad (15.68)$$

By (15.66) and (15.67),

$$\gamma_{k_1-1}^{(N)} - \frac{1}{2}N^{-\frac{1}{2}+w_2} \leq \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) + N^{-\frac{1}{2}+w_2}, \quad \text{i.e.,} \quad \gamma_{k_1-1}^{(N)} \leq \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) + \frac{3}{2}N^{-\frac{1}{2}+w_2}. \quad (15.69)$$

According to Lemma 14.3, there exist $T_5 = T_5(V_p, \kappa, \epsilon_0, \alpha) > 0$ such that when $N > N_0(V_p, \kappa, \epsilon_0, \alpha)$ and $k \in [\frac{1}{2}\alpha N, (1 - \frac{1}{2}\alpha)N]$, then $|\gamma_k^{(N)} - \gamma_k| \leq T_5 N^{-\frac{1}{2}+2\epsilon_0}$. Since $\tilde{\rho}(t) \neq 0$ for $t \in (c, d)$, there exists $s > 0$ depending on V_p, κ, ϵ_0 and α such that $\tilde{\rho}(t) > s$ if $t \in [\gamma_{\alpha N/2}, \gamma_{(1-(\alpha/2))N}]$. If $k \in [\alpha N, (1 - \alpha)N]$, then $k - 1 \in [\frac{1}{2}\alpha N, (1 - \frac{1}{2}\alpha)N]$ and

$$\frac{1}{N} = \int_{\gamma_{k-1}}^{\gamma_k} \tilde{\rho}(t) dt \geq |\gamma_k - \gamma_{k-1}|s,$$

so $|\gamma_k - \gamma_{k-1}| \leq \frac{1}{sN}$ and

$$|\gamma_k^{(N)} - \gamma_{k-1}^{(N)}| \leq |\gamma_k - \gamma_k^{(N)}| + |\gamma_k - \gamma_{k-1}| + |\gamma_{k-1} - \gamma_{k-1}^{(N)}| \leq 2T_5 N^{-\frac{1}{2}+2\epsilon_0} + \frac{1}{sN}. \quad (15.70)$$

So by (15.69) and (15.70), if $N > N_0(V_p, \kappa, \epsilon_0, \alpha, w_1, w_2)$ and $k \in [\alpha N, (1 - \alpha)N]$, then

$$\mathbb{E}^{\tilde{\mu}_s}(\lambda_k) \geq \gamma_{k-1}^{(N)} - \frac{3}{2}N^{-\frac{1}{2}+w_2} \geq \gamma_k^{(N)} - (2T_5 N^{-\frac{1}{2}+2\epsilon_0} + \frac{1}{sN}) - \frac{3}{2}N^{-\frac{1}{2}+w_2} \geq \gamma_k^{(N)} - 2N^{-\frac{1}{2}+w_2}$$

and according to (15.68)

$$|\mathbb{E}^{\tilde{\mu}_s}(\lambda_k) - \gamma_k^{(N)}| \leq 2N^{-\frac{1}{2}+w_2}. \quad (15.71)$$

According to Lemma 13.8, Lemma 14.3 and (15.71), when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, w_1, w_2)$,

$$\begin{aligned} & \mathbb{P}^{\tilde{\mu}_s} \left(\exists k \in [\alpha N, (1 - \alpha)N] \text{ st. } |\lambda_k - \gamma_k| > 6N^{-\frac{1}{2}+w_2} \right) \\ & \leq \mathbb{P}^{\tilde{\mu}_s} \left(\exists k \in [\alpha N, (1 - \alpha)N] \text{ st. } |\lambda_k - \mathbb{E}^{\tilde{\mu}_s}(\lambda_k)| > 3N^{-\frac{1}{2}+w_2} \right) \\ & + \mathbb{P}^{\tilde{\mu}_s} \left(\exists k \in [\alpha N, (1 - \alpha)N] \text{ st. } |\mathbb{E}^{\tilde{\mu}_s}(\lambda_k) - \gamma_k^{(N)}| > 2N^{-\frac{1}{2}+w_2} \right) \\ & + \mathbb{P}^{\tilde{\mu}_s} \left(\exists k \in [\alpha N, (1 - \alpha)N] \text{ st. } |\gamma_k^{(N)} - \gamma_k| > N^{-\frac{1}{2}+w_2} \right) \\ & \leq c_3 \exp(-N^{w_1}) \end{aligned}$$

where $c_3 > 0$ depends on $V_p, \kappa, \epsilon_0, w_1$ and w_2 . Since w_2 can be arbitrarily close to $2\epsilon_0$ and w_2 can be arbitrarily close to w_2 , the theorem is proved. \square

16 From rigidity of scale N^{-1+a_r} to concentration of scale $N^{-1+\frac{a_r}{2}}$

Recall that in Section 5 we defined $\{t_1, t_2, \dots\}$ and $\{P_1, P_2, \dots\}$ satisfying

1. $t_1 > P_1 > 0$.

2. For $k \in \{1, 2, \dots\}$, define $a_k = \frac{1}{2}(\frac{3}{4})^{k-1}$, $P_k = P_1 \times 0.2^{k-1}$ and $t_{k+1} = 2t_k + 1.6P_k$.

Suppose $r \in \{1, 2, \dots\}$. Let $\mathcal{L}(r)$ be the following statement.

$\mathcal{L}(r)$: For any $\alpha \in (0, 1/2)$, there exist $N_0 = N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$ and $i \in [\alpha N, (1 - \alpha)N]$, then

$$\mathbb{P}^{\tilde{\mu}_s}(|\lambda_i - \gamma_i| > N^{-1+a_r+t_r\epsilon_0}) \leq \exp(-N^{P_r\epsilon_0}).$$

Throughout this section we suppose that $\mathcal{L}(r)$ is true. Remember that there is $W > 0$ such that $V_p''(x) \geq -2U_p$ for x in a neighborhood of $[a, b]$.

Recall that $d\nu = \frac{1}{Z_\nu} \exp(-N\beta\mathcal{H}_\nu)d\lambda$ and $d\nu_s = \frac{1}{Z_{\nu_s}} \exp(-N\beta\mathcal{H}_\nu)d\lambda$ are probability measures on $[a, b]^N$ and $\tilde{\Sigma}_N$ respectively with same Hamiltonian

$$\mathcal{H}_\nu = \tilde{\mathcal{H}} + \psi^{(s)} + \sum_{i,j} \psi_{i,j} = \mathcal{H}_{\tilde{\mu}} + \psi^{(s)} + \sum_{i,j} \psi_{i,j} + M \sum_{\alpha=1}^l X_\alpha^2$$

where

- $X_\alpha = N^{-1/2} \sum_{j=1}^N (g_\alpha(\lambda_j) - g_\alpha(\tilde{\gamma}_j))$;
- $\psi^{(s)}(\lambda) = N\theta\left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2\right)$;
- $\psi_{i,j}(\lambda) = \frac{1}{N}\theta\left(\sqrt{\tilde{c}_1 N Q_{i,j}}(\lambda_i - \lambda_j)\right)$;
- $\theta(x) = (x-1)^2 \mathbb{1}_{x>1} + (x+1)^2 \mathbb{1}_{x<-1}$.

So \mathcal{H}_ν is a random variable depending on $V_p, \kappa, \epsilon_0, c_N, M, l, s, \tilde{c}_1, \epsilon$ and N . Remember that M, l, s, \tilde{c}_1 and ϵ are determined by V_p, κ and ϵ_0 since Section 13.2. Thus \mathcal{H}_ν actually depends on $V_p, \kappa, \epsilon_0, c_N$ and N .

Now suppose $\alpha \in (0, 1/2)$, $k \in [\alpha N, (1 - \alpha)N]$ and $M' \in [1, \frac{1}{2}\alpha N] \cap \mathbb{N}$. Define

$$\phi^{(k, M')} = \sum_{\substack{i < j \\ i, j \in I(k, M')}} \theta\left(\frac{N^{1-t_r\epsilon_0}(\lambda_i - \lambda_j)}{|I(k, M')|}\right)$$

where $I(k, M') = [k - M', k + M'] \cap \mathbb{N}$ and $|I(k, M')| = 2M' + 1$.

Suppose $\omega = \omega(N)$ and $\omega_r = \omega_s(N)$ are probability measures on $[a, b]^N$ and $\tilde{\Sigma}_N$ respectively with same Hamiltonian $\mathcal{H}_\omega = \frac{1}{N}\phi^{(k, M')} + \mathcal{H}_\nu$:

$$d(\omega) = \frac{1}{Z_\omega} e^{-\beta N \mathcal{H}_\omega} d\lambda = \frac{1}{Z_\omega} e^{-\beta \phi^{(k, M')}} e^{-\beta N \mathcal{H}_\nu} d\lambda, \quad d(\omega_s) = \frac{1}{Z_{\omega_s}} e^{-\beta N \mathcal{H}_\omega} d\lambda = \frac{1}{Z_{\omega_s}^s} e^{-\beta \phi^{(k, M')}} e^{-\beta N \mathcal{H}_\nu} d\lambda$$

where Z_ω and Z_{ω_s} are normalization constants. So ω and ω_s depend on by $V_p, \kappa, \epsilon_0, N, c_N, t_r, k$ and M' .

Notice that $\mathcal{H}_\omega = \mathcal{H}_1 + \mathcal{H}_2$ where

$$\mathcal{H}_1 = \frac{1}{N} \phi^{(k, M')} - \frac{1}{4N} \sum_{\substack{i < j \\ i, j \in I(k, M')}} \ln |\lambda_i - \lambda_j|, \quad \mathcal{H}_2 = \mathcal{H}_\nu + \frac{1}{4N} \sum_{\substack{i < j \\ i, j \in I(k, M')}} \ln |\lambda_i - \lambda_j|.$$

The following lemma is an analogue of Lemma 3.8 of [6].

Lemma 16.1. *For any $\lambda \in [a, b]^N$ and $\mathbf{v} \in \mathbb{R}^N$,*

$$\begin{aligned} \langle \mathbf{v}, (\nabla^2 \mathcal{H}_2(\lambda)) \mathbf{v} \rangle &\geq \frac{1}{2} \langle \mathbf{v}, (\nabla^2 \mathcal{H}_\nu(\lambda)) \mathbf{v} \rangle - \frac{c_N}{2} U_p \|\mathbf{v}\|^2 \geq 0; \\ \langle \mathbf{v}, (\nabla^2 \mathcal{H}_1(\lambda)) \mathbf{v} \rangle &\geq \frac{1}{8} \frac{N^{1-2t_r \epsilon_0}}{|I(k, M')|^2} \sum_{i, j \in I(k, M')} (v_i - v_j)^2. \end{aligned}$$

Proof. The second inequality in the first statement is trivial thanks to (13.57) and $|c_N - 1| \leq N^{-1+\epsilon_0}$. To prove the first inequality in the first statement we only have to show that

$$\langle \mathbf{v}, \nabla^2 \mathcal{H}_2(\lambda) \mathbf{v} \rangle - \frac{c_N}{4} \sum V_p''(\lambda_i) v_i^2 - \frac{1}{2} \langle \mathbf{v}, \nabla^2 \mathcal{H}_\nu(\lambda) \mathbf{v} \rangle \geq 0$$

since $V_p''(\lambda_i) \geq -2U_p$. On the other hand,

$$\begin{aligned} &\langle \mathbf{v}, \nabla^2 \mathcal{H}_2(\lambda) \mathbf{v} \rangle - \frac{c_N}{4} \sum V_p''(\lambda_i) v_i^2 - \frac{1}{2} \langle \mathbf{v}, \nabla^2 \mathcal{H}_\nu(\lambda) \mathbf{v} \rangle \\ &= \frac{1}{2} \langle \mathbf{v}, \nabla^2 \mathcal{H}_\nu(\lambda) \mathbf{v} \rangle - \frac{c_N}{4} \sum V_p''(\lambda_i) v_i^2 - \frac{1}{4N} \sum_{\substack{i < j \\ i, j \in I(k, M')}} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} \\ &= \frac{1}{2} \left[\frac{1}{N} \sum_{i < j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} + \frac{1}{2} c_N \sum V_p''(\lambda_i) v_i^2 + 2M \sum_{\alpha=1}^l \left[\left(\frac{1}{\sqrt{N}} \sum g'_\alpha(\lambda_j) v_j \right)^2 + X_\alpha \sum \frac{1}{\sqrt{N}} g''_\alpha(\lambda_j) v_j^2 \right] \right. \\ &\quad + \tilde{c}_1 \sum_{i, j} Q_{i, j} \theta''(\sqrt{\tilde{c}_1 N Q_{i, j}} (\lambda_i - \lambda_j)) (v_i - v_j)^2 + 2s \theta' \left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 \right) \|\mathbf{v}\|^2 \\ &\quad \left. + \frac{(2s)^2}{N} \theta'' \left(\frac{s}{N} \sum (\lambda_i - \tilde{\gamma}_i)^2 \right) \left(\sum (\lambda_i - \tilde{\gamma}_i) v_i \right)^2 \right] - \frac{c_N}{4} \sum V_p''(\lambda_i) v_i^2 - \frac{1}{4N} \sum_{\substack{i < j \\ i, j \in I(k, M')}} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} \\ &\geq \frac{1}{2} \left[\frac{1}{2N} \sum_{i < j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2} + 2M \sum_{\alpha=1}^l \left[\left(\frac{1}{\sqrt{N}} \sum g'_\alpha(\lambda_j) v_j \right)^2 + X_\alpha \sum \frac{1}{\sqrt{N}} g''_\alpha(\lambda_j) v_j^2 \right] \right. \\ &\quad \left. + \tilde{c}_1 \sum_{i, j} Q_{i, j} \theta''(\sqrt{\tilde{c}_1 N Q_{i, j}} (\lambda_i - \lambda_j)) (v_i - v_j)^2 + 2s \theta' \left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 \right) \|\mathbf{v}\|^2 \right] \\ &\geq \frac{1}{2} \left[\langle \mathbf{v}, \left(\frac{\tilde{c}_1}{4} \mathcal{Q} + M \sum_{\alpha=1}^l |\mathbf{G}_\alpha\rangle \langle \mathbf{G}_\alpha| - 1 \right) \mathbf{v} \rangle + (1 - C(M, l) \Delta) \|\mathbf{v}\|^2 + \|\mathbf{v}\|^2 4s \left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 - 1 \right) \mathbf{1}_{\left(\frac{s}{N} \sum_{i=1}^N (\lambda_i - \tilde{\gamma}_i)^2 > 1 \right)} \right] \end{aligned}$$

where $C(M, l) = 2M \sum_{\alpha=1}^l (\|g''_\alpha\|_\infty^2 + \|g'_\alpha\|_\infty \|g''_\alpha\|_\infty)$. Here we used (11.48) and the same argument as (12.53). According to Lemma 12.2, $\langle \mathbf{v}, \nabla^2 \mathcal{H}_2(\lambda) \mathbf{v} \rangle - \frac{c_N}{4} \sum V_p''(\lambda_i) v_i^2 - \frac{1}{2} \langle \mathbf{v}, \nabla^2 \mathcal{H}_\nu(\lambda) \mathbf{v} \rangle$ is nonnegative for all $\lambda \in [a, b]^N$ and $\mathbf{v} \in \mathbb{R}^N$, so the first statement is proved.

To prove the second statement, we have by direct computation that

$$\langle \mathbf{v}, (\nabla^2 \mathcal{H}_1(\lambda)) \mathbf{v} \rangle = \frac{1}{N} \sum_{\substack{i < j \\ i, j \in I(k, M')}} \theta'' \left(\frac{N^{1-t_r \epsilon_0} (\lambda_i - \lambda_j)}{|I(k, M')|} \right) \frac{N^{2-2t_r \epsilon_0}}{|I(k, M')|^2} (v_i - v_j)^2 + \frac{1}{4N} \sum_{\substack{i < j \\ i, j \in I(k, M')}} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2}.$$

Noticing that

$$\theta'' \left(\frac{N^{1-t_r \epsilon_0} (\lambda_i - \lambda_j)}{|I(k, M')|} \right) \frac{N^{2-2t_r \epsilon_0}}{|I(k, M')|^2} + \frac{1}{4(\lambda_i - \lambda_j)^2} \geq \frac{1}{4} \left(\theta'' \left(\frac{N^{1-t_r \epsilon_0} (\lambda_i - \lambda_j)}{|I(k, M')|} \right) \frac{N^{2-2t_r \epsilon_0}}{|I(k, M')|^2} + \frac{1}{(\lambda_i - \lambda_j)^2} \right) \geq \frac{N^{2-2t_r \epsilon_0}}{4|I(k, M')|^2},$$

we have

$$\langle \mathbf{v}, (\nabla^2 \mathcal{H}_1(\lambda)) \mathbf{v} \rangle \geq \frac{N^{1-2t_r \epsilon_0}}{4|I(k, M')|^2} \sum_{\substack{i < j \\ i, j \in I(k, M')}} (v_i - v_j)^2 = \frac{N^{1-2t_r \epsilon_0}}{8|I(k, M')|^2} \sum_{i, j \in I(k, M')} (v_i - v_j)^2.$$

□

Lemma 16.2. Suppose $N \geq 1$, $k \in [\alpha N, (1 - \alpha)N]$ and $M' \in [1, \frac{1}{2}\alpha N] \cap \mathbb{N}$. Suppose $F : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function and $f(\lambda) = F(\sum_{i \in I(k, M')} v_i \lambda_i)$ is a function on $\tilde{\Sigma}_N$ where $\sum_{i \in I(k, M')} v_i = 0$. Then for $\omega_s = \omega_s(k, M')$ we have

$$\int_{\tilde{\Sigma}_N} f^2(\lambda) \ln(f^2(\lambda)) d\omega_s - \int_{\tilde{\Sigma}_N} f^2(\lambda) d\omega_s \ln \left(\int_{\tilde{\Sigma}_N} f^2(\lambda) d\omega_s \right) \leq \frac{8|I(k, M')|}{N^{2-2t_r \epsilon_0}} \int_{\tilde{\Sigma}_N} |\nabla f|^2 d\omega_s.$$

Using the argument of Lemma 4.4 of [14], one can prove Lemma 16.2 in the same way as Lemma 3.9 of [6].

The following lemma is an analogue of Corollary 3.10 of [6].

Lemma 16.3. Suppose $\alpha \in (0, 1/2)$, $N \geq 1$, $k \in [\alpha N, (1 - \alpha)N]$ and $M' \in [1, \frac{1}{2}\alpha N] \cap \mathbb{N}$. If $\sum_{i \in I(k, M')} v_i = 0$ and $w > 0$, then for $\omega_s = \omega_s(k, M')$ we have

$$\mathbb{P}^{\omega_s} \left(\left| \sum_{i \in I(k, M')} v_i \lambda_i - \mathbb{E}^{\omega_s} \left(\sum_{i \in I(k, M')} v_i \lambda_i \right) \right| > w \right) \leq 2 \exp \left(- \frac{w^2}{\sum_{i \in I(k, M')} v_i^2 \frac{N^{2-2t_r \epsilon_0}}{8|I(k, M')|}} \right).$$

Proof. Suppose \bar{P} is a probability measure on \mathbb{R} such that $\bar{P}(A) = \omega_s(\lambda \mid \sum_{i \in I(k, M')} v_i \lambda_i \in A)$. By the classic argument of measure theory we have that for every measurable function $h : \mathbb{R} \rightarrow \mathbb{R}$,

$$\int_{\tilde{\Sigma}_N} h \left(\sum_{i \in I(k, M')} v_i \lambda_i \right) d\omega_s = \int_{\mathbb{R}} h(x) d\bar{P}$$

as long as the left hand side or the right hand side makes sense. In particular $\mathbb{E}^{\omega_s}(\sum_{i \in I(k, M')} v_i \lambda_i) = \mathbb{E}^{\bar{P}}(x)$. If $F : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function and $f(\lambda) = F(\sum_{i \in I(k, M')} v_i \lambda_i)$, then by Lemma 16.2,

$$\begin{aligned} \int F^2(x) \ln F^2(x) d\bar{P} - \int F^2(x) d\bar{P} \ln \left(\int F^2(x) d\bar{P} \right) &\leq \frac{8|I(k, M')|}{N^{2-2t_r \epsilon_0}} \int_{\tilde{\Sigma}_N} |\nabla f|^2 d\omega_s \\ &= \frac{8|I(k, M')| \sum_{i \in I(k, M')} v_i^2}{N^{2-2t_r \epsilon_0}} \int |F'(x)|^2 d\bar{P}. \end{aligned}$$

Since $x \mapsto x$ is a Lipschitz function on \mathbb{R}^N with Lipschitz constant 1, according to Lemma 13.2 we have that

$$\mathbb{P}^{\omega_s} \left(\left| \sum_{i \in I(k, M')} v_i \lambda_i - \mathbb{E}^{\omega_s} \left(\sum_{i \in I(k, M')} v_i \lambda_i \right) \right| > w \right) = \bar{P} \left(\left| x - \mathbb{E}^{\bar{P}} x \right| > w \right) \leq 2 \exp \left(- \frac{w^2}{\sum_{i \in I(k, M')} v_i^2} \frac{N^{2-2t_r \epsilon_0}}{8|I(k, M')|} \right).$$

□

For $\lambda \in [a, b]^N$, $k \in [\alpha N, (1 - \alpha)N]$, and $N' \in [1, \frac{1}{2}\alpha N] \cap \mathbb{N}$, define

$$\lambda_k^{[N']} = \frac{1}{2N' + 1} \sum_{i=k-N'}^{k+N'} \lambda_i.$$

The following lemma is an analogue of Lemma 3.14 of [6].

Lemma 16.4. *Suppose $\alpha \in (0, 1/2)$, $N \geq 1$, $k \in [\alpha N, (1 - \alpha)N]$, $1 \leq M'' \leq M' \leq \frac{1}{2}\alpha N$. Here M' and M'' are in \mathbb{N} . If $w > 0$, then for $\omega_s = \omega_s(k, M')$ we have*

$$\begin{aligned} \mathbb{P}^{\omega_s} \left(\left| \lambda_k^{[M'']} - \lambda_k^{[M']} - \mathbb{E}^{\omega_s} (\lambda_k^{[M'']} - \lambda_k^{[M']}) \right| > \frac{w}{N^{1-t_r \epsilon_0}} \sqrt{\frac{M'}{M''}} \right) &\leq 2 \exp \left(- \frac{w^2}{12} \right); \\ \mathbb{P}^{\omega_s} \left(\left| \lambda_k - \lambda_k^{[M']} - \mathbb{E}^{\omega_s} (\lambda_k - \lambda_k^{[M']}) \right| > \frac{w}{N^{1-t_r \epsilon_0}} \sqrt{M'} \right) &\leq 2 \exp \left(- \frac{w^2}{24} \right). \end{aligned}$$

Proof. $\lambda_k^{[M'']} - \lambda_k^{[M']}$ has the form $\sum_{i=k-M'}^{k+M'} v_i \lambda_i$ with $\sum v_i = 0$. Furthermore,

$$\sum_{i=k-M'}^{k+M'} v_i^2 = (2M'' + 1) \left(\frac{1}{2M' + 1} - \frac{1}{2M'' + 1} \right)^2 + 2(M' - M'') \left(\frac{1}{2M' + 1} \right)^2 \leq \frac{1}{2M''}.$$

This together with Lemma 16.3 prove the first statement. The second statement can be proved in the same way with $\sum v_i^2 \leq 1$. □

Lemma 16.5 (Pinsker's inequality). *Suppose ν_1 and ν_2 are probability measures on a same probability space and ν_1 is absolutely continuous with respect to ν_2 . Then we have*

$$\sup_{A \text{ is measurable}} |\nu_1(A) - \nu_2(A)| \leq \left(\frac{1}{2} \int \ln \left(\frac{d\nu_1}{d\nu_2} \right) d\nu_1 \right)^{1/2} = \left(\frac{1}{2} \int \frac{d\nu_1}{d\nu_2} \ln \left(\frac{d\nu_1}{d\nu_2} \right) d\nu_2 \right)^{1/2}$$

where $\frac{d\nu_1}{d\nu_2}$ is the Radon-Nikodym derivative.

Proof. This lemma can be found in many books. □

Lemma 16.6. *Suppose $P_r > 20$ (thus $t_r > 3$). Suppose $\mathcal{L}(r)$ holds and $\alpha \in (0, 1/2)$. There exists $N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$, $k \in [\alpha N, (1 - \alpha)N]$, $2N^{a_r} \leq M' \leq \frac{1}{2}\alpha N$, then for $\omega_s = \omega_s(k, M')$ we have*

$$\left| \int \frac{d\omega_s}{d\nu_s} \ln \left(\frac{d\omega_s}{d\nu_s} \right) d\nu_s \right| \leq \exp(-N^{0.9P_r \epsilon_0}).$$

Proof. There exists $s = s(V_p, \kappa, \epsilon_0, \alpha) > 0$ such that if $N \geq 1$ and $x \in [\gamma_{\frac{1}{2}\alpha N}, \gamma_{(1-\frac{1}{2}\alpha)N}]$ then $\tilde{\rho}(x) > s$. There exists $N_1 = N_1(a_r, \alpha)$ such that if $N > N_1$ then $[k - 2N^{a_r}, k + 2N^{a_r}] \subset [\frac{1}{2}\alpha N, (1 - \frac{1}{2}\alpha)N]$. So when $N > N_1(a_r, \alpha)$,

$$s|\gamma_{k+M'} - \gamma_{k-M'}| \leq \int_{\gamma_{k-M'}}^{\gamma_{k+M'}} \tilde{\rho}(x) dx = \frac{2M'}{N}.$$

and $|\gamma_{k+M'} - \gamma_{k-M'}| \leq \frac{2}{s} \frac{M'}{N}$.

Suppose $\lambda \in \tilde{\Sigma}_N$. If

$$\phi^{(k, M')} = \sum_{\substack{i < j \\ i, j \in I(k, M')}} \theta\left(\frac{N^{1-t_r\epsilon_0}(\lambda_i - \lambda_j)}{|I(k, M')|}\right) \neq 0,$$

then $\lambda_{k+M'} - \lambda_{k-M'} \geq |I(k, M')|N^{-1+t_r\epsilon_0} > 2M'N^{-1+t_r\epsilon_0}$. If $N > N_2(V_p, \kappa, \epsilon_0, a_r, \alpha)$, then

$$\begin{aligned} |\lambda_{k-M'} - \gamma_{k-M'}| + |\lambda_{k+M'} - \gamma_{k+M'}| &\geq |\lambda_{k+M'} - \lambda_{k-M'}| - |\gamma_{k+M'} - \gamma_{k-M'}| \geq 2M'N^{-1+t_r\epsilon_0} - \frac{2}{s} \frac{M'}{N} \\ &\geq M'N^{-1+t_r\epsilon_0} \end{aligned}$$

therefore at least one of $|\lambda_{k-M'} - \gamma_{k-M'}|$, $|\lambda_{k+M'} - \gamma_{k+M'}|$ is no less than $\frac{1}{2}M'N^{-1+t_r\epsilon_0}$ and thus no less than $N^{-1+a_r+t_r\epsilon_0}$.

According to $\mathcal{L}(r)$, there exist $N_3 = N_3(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_3$, then

$$\begin{aligned} \mathbb{P}^{\tilde{\mu}_s}(\phi^{(k, M')} \neq 0) &\leq \mathbb{P}^{\tilde{\mu}_s}(|\lambda_{k+M'} - \lambda_{k-M'}| \geq N^{-1+t_r\epsilon_0}|I(k, M')|) \\ &\leq \mathbb{P}^{\tilde{\mu}_s}(|\lambda_{k-M'} - \gamma_{k-M'}| > N^{-1+a_r+t_r\epsilon_0}) + \mathbb{P}^{\tilde{\mu}_s}(|\lambda_{k+M'} - \gamma_{k+M'}| > N^{-1+a_r+t_r\epsilon_0}) \\ &\leq 2\exp(-N^{P_r\epsilon_0}). \end{aligned}$$

According to Lemma 13.7, there are $N_4 = N_4(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_4$, then

$$\mathbb{P}^{\nu_s}(\phi^{(k, M')} \neq 0) \leq \mathbb{P}^{\nu_s}(|\lambda_{k+M'} - \lambda_{k-M'}| \geq N^{-1+t_r\epsilon_0}|I(k, M')|) \leq \exp(-N^{0.99P_r\epsilon_0}) < \frac{1}{2}. \quad (16.72)$$

So when $N > N_4$,

$$\int_{\tilde{\Sigma}_N} \exp(-\beta\phi^{(k, M')}) d\nu_s \geq \mathbb{P}^{\nu_s}(\phi^{(k, M')} = 0) \geq \frac{1}{2} \quad (16.73)$$

and

$$\mathbb{P}^{\omega_s}(A) = \frac{\int_A \exp(-\beta\phi^{(k, M')}) d\nu_s}{\int_{\tilde{\Sigma}_N} \exp(-\beta\phi^{(k, M')}) d\nu_s} \leq 2 \int_A d\nu_s = 2\mathbb{P}^{\nu_s}(A) \quad \forall A \subset \tilde{\Sigma}_N.$$

Suppose f is a function on $\tilde{\Sigma}_N$ defined by $f(\lambda) = \frac{d\omega_s}{d\nu_s} = \frac{\exp(-\beta\phi^{(k,M')})}{\int_{\tilde{\Sigma}_N} \exp(-\beta\phi^{(k,M')}) d\nu_s}$. By (16.73), $\|f\|_\infty \leq 2$ when $N > N_4$. If $i \notin I(k, M')$, $\frac{\partial\phi^{(k,M')}}{\partial\lambda_i} = 0$. If $i \in I(k, M')$, then

$$\begin{aligned} \left| \frac{\partial\phi^{(k,M')}}{\partial\lambda_i} \right| &= \left| \frac{N^{1-t_r\epsilon_0}}{|I(k, M')|} \left(\sum_{\substack{i < j \\ j \in I(k, M')}} \theta' \left(\frac{N^{1-t_r\epsilon_0}(\lambda_i - \lambda_j)}{|I(k, M')|} \right) - \sum_{\substack{i > j \\ j \in I(k, M')}} \theta' \left(\frac{N^{1-t_r\epsilon_0}(\lambda_i - \lambda_j)}{|I(k, M')|} \right) \right) \right| \\ &\leq N^{1-t_r\epsilon_0} \theta' \left(\frac{N^{1-t_r\epsilon_0}(\lambda_{k+M'} - \lambda_{k-M'})}{|I(k, M')|} \right) \end{aligned}$$

Therefore when $N > N_4$

$$\begin{aligned} \|\nabla \sqrt{f}\|^2 &= \left\| -\frac{\beta\sqrt{f}}{2} \left(\frac{\partial\phi^{(k,M')}}{\partial\lambda_1}, \dots, \frac{\partial\phi^{(k,M')}}{\partial\lambda_N} \right) \right\|^2 = \frac{\beta^2 f(\lambda)}{4} \sum_{i=1}^N \left| \frac{\partial\phi^{(k,M')}}{\partial\lambda_i} \right|^2 \\ &\leq \frac{\beta^2}{2} (2M' + 1) N^{2-2t_r\epsilon_0} \left| \theta' \left(\frac{N^{1-t_r\epsilon_0}(\lambda_{k+M'} - \lambda_{k-M'})}{|I(k, M')|} \right) \right|^2. \end{aligned}$$

According to (13.57), Lemma 13.1 and the fact that $\int_{\tilde{\Sigma}_N} f d\nu_s = 1$, when $N > N_5(V_p, \kappa, \epsilon_0, \alpha, a_r)$,

$$\begin{aligned} \mathbb{E}^{\nu_s}(f \ln f) &\leq \frac{1}{2\beta N} \int_{\tilde{\Sigma}_N} \|\nabla \sqrt{f}\|^2 d\nu_s \leq \frac{\beta}{4} (2M' + 1) N^{1-2t_r\epsilon_0} \mathbb{E}^{\nu_s} \left[\theta' \left(\frac{N^{1-t_r\epsilon_0}(\lambda_{k+M'} - \lambda_{k-M'})}{|I(k, M')|} \right)^2 \right] \\ &\leq N^2 \mathbb{E}^{\nu_s} \left[\theta' \left(\frac{N^{1-t_r\epsilon_0}(\lambda_{k+M'} - \lambda_{k-M'})}{|I(k, M')|} \right)^2 \right]. \end{aligned}$$

According to (16.72), when $N > N_6(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\mathbb{P}^{\nu_s} \left(\theta' \left(\frac{N^{1-t_r\epsilon_0}(\lambda_{k+M'} - \lambda_{k-M'})}{|I(k, M')|} \right) \neq 0 \right) \leq \exp(-N^{0.99P_r\epsilon_0}).$$

Since $|\theta'(x)| \leq 2|x|$, we have that for $N > N_7(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\begin{aligned} \mathbb{E}^{\nu_s} \left[\theta' \left(\frac{N^{1-t_r\epsilon_0}(\lambda_{k+M'} - \lambda_{k-M'})}{|I(k, M')|} \right)^2 \right] &\leq 4 \left(\frac{N^{1-t_r\epsilon_0}(\lambda_{k+M'} - \lambda_{k-M'})}{|I(k, M')|} \right)^2 \exp(-N^{0.99P_r\epsilon_0}) \\ &\leq (|a| + |b|)^2 N^{2-2t_r\epsilon_0-2a_r} \exp(-N^{0.99P_r\epsilon_0}) \end{aligned}$$

and

$$\mathbb{E}^{\nu_s}(f \ln f) \leq (|a| + |b|)^2 N^{4-2t_r\epsilon_0-2a_r} \exp(-N^{0.99P_r\epsilon_0}) \leq \exp(-N^{0.9P_r\epsilon_0}).$$

□

The following lemma is an analogue of Lemma 3.15 of [6].

Lemma 16.7. *Suppose $P_r > 20$ (thus $t_r > 3$). Suppose $\mathcal{L}(r)$ holds and $\alpha \in (0, 1/2)$. There exists $N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$, $k \in [\alpha N, (1 - \alpha)N]$, $2N^{a_r} \leq M' \leq \frac{1}{2}\alpha N$, then for all $j \in [1, N]$ and $\omega_s = \omega_s(k, M')$ we have*

$$|\mathbb{E}^{\nu_s}(\lambda_j) - \mathbb{E}^{\omega_s}(\lambda_j)| \leq \exp(-N^{0.44P_r\epsilon_0}).$$

Proof of Lemma 16.7. If g is a bounded function on $\tilde{\Sigma}_N$, then by the classic argument of measure theory,

$$\left| \int g d\nu_s - \int g d\omega_s \right| \leq \|g\|_\infty \sup_A |\nu_s(A) - \omega_s(A)|$$

where A runs over measurable subsets of $\tilde{\Sigma}_N$. So by Lemma 16.5, for $N \geq 1$ and $j \in [1, N]$,

$$|\mathbb{E}^{\nu_s}(\lambda_j) - \mathbb{E}^{\omega_s}(\lambda_j)| \leq \left(\frac{1}{2} \int \frac{d\omega_s}{d\nu_s} \ln \left(\frac{d\omega_s}{d\nu_s} \right) d\nu_s \right)^{1/2} (|a| + |b|)$$

Applying Lemma 16.6 we complete the proof. \square

Lemma 16.8. *Suppose $P_r > 20$ (thus $t_r > 3$). Suppose $\mathcal{L}(r)$ holds and $\alpha \in (0, 1/2)$. There exists $N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$, $k \in [\alpha N, (1 - \alpha)N]$, $2N^{a_r} \leq M' \leq \frac{1}{2}\alpha N$ and $1 \leq M'' \leq M'$ then we have*

$$\mathbb{P}^{\nu_s} \left(\left| \lambda_k^{[M'']} - \lambda_k^{[M']} - \mathbb{E}^{\nu_s}(\lambda_k^{[M'']} - \lambda_k^{[M']}) \right| > N^{-1+t_r\epsilon_0+0.5P_r\epsilon_0} \sqrt{\frac{M'}{M''}} \right) \leq \exp(-N^{0.437P_r\epsilon_0});$$

$$\mathbb{P}^{\nu_s} \left(\left| \lambda_k - \lambda_k^{[M']} - \mathbb{E}^{\nu_s}(\lambda_k - \lambda_k^{[M']}) \right| > N^{-1+t_r\epsilon_0+0.5P_r\epsilon_0} \sqrt{M'} \right) \leq \exp(-N^{0.43P_r\epsilon_0}).$$

Proof of Lemma 16.8. By Lemma 16.7, when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\begin{aligned} & |\mathbb{E}^{\nu_s}(\lambda_k^{[M'']} - \lambda_k^{[M']}) - \mathbb{E}^{\omega_s}(\lambda_k^{[M'']} - \lambda_k^{[M']})| \quad (\text{Here } \omega_s = \omega_s(k, M')) \\ &= \left| \frac{1}{2M''+1} \sum_{i=k-M''}^{k+M''} (\mathbb{E}^{\nu_s} \lambda_i - \mathbb{E}^{\omega_s} \lambda_i) + \frac{1}{2M'+1} \sum_{i=k-M'}^{k+M'} (\mathbb{E}^{\nu_s} \lambda_i - \mathbb{E}^{\omega_s} \lambda_i) \right| \\ &\leq 2 \exp(-N^{0.44P_r\epsilon_0}). \end{aligned}$$

By Lemma 16.4, when $N > N_0$,

$$\begin{aligned} & \mathbb{P}^{\omega_s} \left(\left| \lambda_k^{[M'']} - \lambda_k^{[M']} - \mathbb{E}^{\nu_s}(\lambda_k^{[M'']} - \lambda_k^{[M']}) \right| > N^{-1+t_r\epsilon_0+0.5P_r\epsilon_0} \sqrt{\frac{M'}{M''}} \right) \\ &\leq \mathbb{P}^{\omega_s} \left(\left| \lambda_k^{[M'']} - \lambda_k^{[M']} - \mathbb{E}^{\omega_s}(\lambda_k^{[M'']} - \lambda_k^{[M']}) \right| > N^{-1+t_r\epsilon_0+0.49P_r\epsilon_0} \sqrt{\frac{M'}{M''}} \right) \\ &\leq 2 \exp\left(-\frac{N^{0.98P_r\epsilon_0}}{12}\right) \end{aligned}$$

By Lemma 16.5 and Lemma 16.6, when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\begin{aligned} & \mathbb{P}^{\nu_s} \left(\left| \lambda_k^{[M'']} - \lambda_k^{[M']} - \mathbb{E}^{\nu_s}(\lambda_k^{[M'']} - \lambda_k^{[M']}) \right| > N^{-1+t_r\epsilon_0+0.5P_r\epsilon_0} \sqrt{\frac{M'}{M''}} \right) \\ &\leq \mathbb{P}^{\omega_s} \left(\left| \lambda_k^{[M'']} - \lambda_k^{[M']} - \mathbb{E}^{\nu_s}(\lambda_k^{[M'']} - \lambda_k^{[M']}) \right| > N^{-1+t_r\epsilon_0+0.5P_r\epsilon_0} \sqrt{\frac{M'}{M''}} \right) + \left(\frac{1}{2} \int \frac{d\omega_s}{d\nu_s} \ln \left(\frac{d\omega_s}{d\nu_s} \right) d\nu_s \right)^{1/2} \\ &\leq 2 \exp\left(-\frac{N^{0.98P_r\epsilon_0}}{12}\right) + \exp(-N^{0.44P_r\epsilon_0}) \\ &\leq 2 \exp(-N^{0.44P_r\epsilon_0}) \end{aligned}$$

So the first statement is proved. The second statement can be proved in the same way. \square

If $x \geq 0$ is not an integer, then we define $\lambda_k^{[x]}$ to be $\lambda_k^{\lfloor x \rfloor}$ where $\lfloor x \rfloor$ is the largest integer no more than x .

The following lemma is an analogue of Lemma 3.17 of [6].

Lemma 16.9. *Suppose $P_r > 20$ (thus $t_r > 3$). Suppose $\mathcal{L}(r)$ holds and $\alpha \in (0, 1/2)$. There exists $N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$ and $k \in [\alpha N, (1 - \alpha)N]$ then we have*

$$\mathbb{P}^{\nu_s} \left(\left| \lambda_k - \lambda_k^{[\alpha N/2]} - \mathbb{E}^{\nu_s}(\lambda_k - \lambda_k^{[\alpha N/2]}) \right| > N^{-1 + \frac{a_r}{2} + t_r \epsilon_0 + 0.6 P_r \epsilon_0} \right) \leq \exp(N^{-0.42 P_r \epsilon_0}).$$

Proof.

$$\begin{aligned} & \left| \lambda_k - \lambda_k^{[\alpha N/2]} - \mathbb{E}^{\nu_s}(\lambda_k - \lambda_k^{[\alpha N/2]}) \right| \\ & \leq \left| \lambda_k - \lambda_k^{[3N^{a_r}]} - \mathbb{E}^{\nu_s}(\lambda_k - \lambda_k^{[3N^{a_r}]}) \right| + \left| \lambda_k^{[\alpha N/2]} - \lambda_k^{[3N^{a_r}]} - \mathbb{E}^{\nu_s}(\lambda_k^{[\alpha N/2]} - \lambda_k^{[3N^{a_r}]}) \right| \\ & := I + II \end{aligned}$$

According to Lemma 16.8, when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\mathbb{P}^{\nu_s}(I > \sqrt{3}N^{-1 + t_r \epsilon_0 + 0.5 P_r \epsilon_0 + \frac{a_r}{2}}) \leq \mathbb{P}^{\nu_s}(I > N^{-1 + t_r \epsilon_0 + 0.5 P_r \epsilon_0} \sqrt{[3N^{a_r}]}) \leq \exp(-N^{0.43 P_r \epsilon_0}).$$

Suppose $r \in (0, a_r)$ and $q \in \mathbb{N}$ depending on ϵ_0 and a_r such that $1 - r < a_r + qr < 1$, then

$$II \leq \sum_{l=0}^{q-1} W_l + Z$$

where

$$W_l = \left| \lambda_k^{[3N^{a_r + (l+1)r}]} - \lambda_k^{[3N^{a_r + lr}]} - \mathbb{E}^{\nu_s}(\lambda_k^{[3N^{a_r + (l+1)r}]} - \lambda_k^{[3N^{a_r + lr}]}) \right|$$

and

$$Z = \left| \lambda_k^{[\alpha N/2]} - \lambda_k^{[3N^{a_r + qr}]} - \mathbb{E}^{\nu_s}(\lambda_k^{[\alpha N/2]} - \lambda_k^{[3N^{a_r + qr}]}) \right|.$$

According to Lemma 16.8, when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\begin{aligned} \mathbb{P}^{\nu_s}(W_l > 2N^{-1 + t_r \epsilon_0 + 0.5 P_r \epsilon_0 + \frac{r}{2}}) & \leq \exp(N^{-0.43 P_r \epsilon_0}) \quad \forall l \in [0, q-1]; \\ \mathbb{P}^{\nu_s}(Z > 2N^{-1 + t_r \epsilon_0 + 0.5 P_r \epsilon_0 + \frac{r}{2}}) & \leq \exp(N^{-0.43 P_r \epsilon_0}) \end{aligned}$$

So we have for $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\begin{aligned} & \mathbb{P}^{\nu_s} \left(\left| \lambda_k - \lambda_k^{[\alpha N/2]} - \mathbb{E}^{\nu_s}(\lambda_k - \lambda_k^{[\alpha N/2]}) \right| > N^{-1 + \frac{a_r}{2} + t_r \epsilon_0 + 0.6 P_r \epsilon_0} \right) \\ & \leq \mathbb{P}^{\nu_s} \left(I > \sqrt{3}N^{-1 + t_r \epsilon_0 + 0.5 P_r \epsilon_0 + \frac{a_r}{2}} \right) + \mathbb{P}^{\nu_s} \left(\sum_{l=0}^{q-1} W_l + Z > 2(q+1)N^{-1 + \frac{r}{2} + t_r \epsilon_0 + 0.5 P_r \epsilon_0} \right) \\ & \leq (q+2) \exp(N^{-0.43 P_r \epsilon_0}) \\ & \leq \exp(N^{-0.42 P_r \epsilon_0}). \end{aligned}$$

□

The following lemma is an analogue of Lemma 3.12 of [6].

Theorem 16.10. *Suppose $P_r > 20$ (thus $t_r > 3$). Suppose $\mathcal{L}(r)$ holds and $\alpha \in (0, 1/2)$. There exists $N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$ and $k \in [\alpha N, (1 - \alpha)N]$ then we have*

$$\mathbb{P}^{\tilde{\mu}_s} \left(\left| \lambda_k - \mathbb{E}^{\tilde{\mu}_s} \lambda_k \right| > N^{-1 + \frac{a_r}{2} + (t_r + 0.7P_r)\epsilon_0} \right) \leq \exp(-N^{0.2P_r\epsilon_0})$$

Proof. Notice that $\left| \lambda_k - \mathbb{E}^{\nu_s} \lambda_k \right| \leq \left| \lambda_k - \lambda_k^{[\alpha N/2]} - \mathbb{E}^{\nu_s} (\lambda_k - \lambda_k^{[\alpha N/2]}) \right| + \left| \lambda_k^{[\alpha N/2]} - \mathbb{E}^{\nu_s} \lambda_k^{[\alpha N/2]} \right|$.

Suppose $G(\lambda)$ is a function on $\tilde{\Sigma}_N$ defined by $G(\lambda) = \lambda_k^{[\alpha N/2]}$. Then it is easy to check that G is a Lipschitz function with Lipschitz constant $\frac{1}{\sqrt{2[\alpha N/2] + 1}}$. By Lemma 13.3 and (13.57),

$$\begin{aligned} \mathbb{P}^{\nu_s} \left(\left| \lambda_k^{[\alpha N/2]} - \mathbb{E}^{\nu_s} \lambda_k^{[\alpha N/2]} \right| > N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.5P_r\epsilon_0} \right) &\leq 2 \exp \left(-N^{-2 + a_r + 2t_r\epsilon_0 + P_r\epsilon_0} 2\beta N (1 + 2\lfloor \frac{\alpha N}{2} \rfloor) \right) \\ &\leq 2 \exp \left(-\alpha\beta N^{a_r + 2t_r\epsilon_0 + P_r\epsilon_0} \right) \end{aligned}$$

when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$

According to Lemma 16.9, when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\mathbb{P}^{\nu_s} \left(\left| \lambda_k - \lambda_k^{[\alpha N/2]} - \mathbb{E}^{\nu_s} (\lambda_k - \lambda_k^{[\alpha N/2]}) \right| > N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.6P_r\epsilon_0} \right) \leq \exp(-N^{0.42P_r\epsilon_0}).$$

Thus when $N > N_0$,

$$\mathbb{P}^{\nu_s} \left(\left| \lambda_k - \mathbb{E}^{\nu_s} \lambda_k \right| > 2N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.6P_r\epsilon_0} \right) \leq 2 \exp \left(-\alpha\beta N^{a_r + 2t_r\epsilon_0 + P_r\epsilon_0} \right) + \exp(-N^{0.42P_r\epsilon_0}) \leq 2 \exp(-N^{0.42P_r\epsilon_0}).$$

By Lemma 13.7 and $P_r > 20$, when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\mathbb{P}^{\tilde{\mu}_s} \left(\left| \lambda_k - \mathbb{E}^{\nu_s} \lambda_k \right| > 2N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.6P_r\epsilon_0} \right) \leq \exp(-N^{0.2P_r\epsilon_0})$$

and

$$\begin{aligned} |\mathbb{E}^{\tilde{\mu}_s} \lambda_k - \mathbb{E}^{\nu_s} \lambda_k| &= \left| \int_{\tilde{\Sigma}_N \cap \{ \lambda \mid |\lambda_k - \mathbb{E}^{\nu_s} \lambda_k| > 2N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.6P_r\epsilon_0} \}} (\lambda_k - \mathbb{E}^{\nu_s} \lambda_k) d\tilde{\mu}_s \right. \\ &\quad \left. + \int_{\tilde{\Sigma}_N \cap \{ \lambda \mid |\lambda_k - \mathbb{E}^{\nu_s} \lambda_k| \leq 2N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.6P_r\epsilon_0} \}} (\lambda_k - \mathbb{E}^{\nu_s} \lambda_k) d\tilde{\mu}_s \right| \\ &\leq (2|a| + 2|b|) \mathbb{P}^{\tilde{\mu}_s} \left(\left| \lambda_k - \mathbb{E}^{\nu_s} \lambda_k \right| > 2N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.6P_r\epsilon_0} \right) + 2N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.6P_r\epsilon_0} \\ &\leq (2|a| + 2|b|) \exp(-N^{0.2P_r\epsilon_0}) + 2N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.6P_r\epsilon_0} \\ &\leq 3N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.6P_r\epsilon_0}. \end{aligned}$$

Therefore when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\mathbb{P}^{\tilde{\mu}_s} \left(\left| \lambda_k - \mathbb{E}^{\tilde{\mu}_s} \lambda_k \right| > N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.7P_r\epsilon_0} \right) \leq \mathbb{P}^{\tilde{\mu}_s} \left(\left| \lambda_k - \mathbb{E}^{\nu_s} \lambda_k \right| > 2N^{-1 + \frac{a_r}{2} + t_r\epsilon_0 + 0.6P_r\epsilon_0} \right) \leq \exp(-N^{0.2P_r\epsilon_0}).$$

□

Corollary 16.11. Suppose $P_r > 20$ (thus $t_r > 3$). Suppose $\mathcal{L}(r)$ is true and $\alpha \in (0, 1/2)$. There exists $N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$ and $k \in [\alpha N, (1 - \alpha)N]$ then

$$|\mathbb{E}^{\nu_s} \lambda_k - \mathbb{E}^{\tilde{\mu}_s} \lambda_k| \leq 2N^{-1 + \frac{\alpha_r}{2} + (t_r + 0.7P_r)\epsilon_0}$$

and

$$\mathbb{P}^{\nu_s} \left(\left| \lambda_k - \mathbb{E}^{\nu_s} \lambda_k \right| > 3N^{-1 + \frac{\alpha_r}{2} + (t_r + 0.7P_r)\epsilon_0} \right) \leq \exp(-N^{0.19P_r\epsilon_0}).$$

Proof. According to Lemma 13.7 and Theorem 16.10, for any $\alpha \in (0, 1/2)$, There exists $N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$ and $k \in [\alpha N, (1 - \alpha)N]$ then we have

$$\mathbb{P}^{\nu_s} \left(\left| \lambda_k - \mathbb{E}^{\tilde{\mu}_s} \lambda_k \right| > N^{-1 + \frac{\alpha_r}{2} + (t_r + 0.7P_r)\epsilon_0} \right) \leq \exp(-N^{0.19P_r\epsilon_0}).$$

So

$$\begin{aligned} |\mathbb{E}^{\nu_s} \lambda_k - \mathbb{E}^{\tilde{\mu}_s} \lambda_k| &\leq \int_{\tilde{\Sigma}_N} |\lambda_k - \mathbb{E}^{\tilde{\mu}_s} \lambda_k| \mathbb{1}_{(|\lambda_k - \mathbb{E}^{\tilde{\mu}_s} \lambda_k| > N^{-1 + \frac{\alpha_r}{2} + (t_r + 0.7P_r)\epsilon_0})} d\nu_s \\ &\quad + \int_{\tilde{\Sigma}_N} |\lambda_k - \mathbb{E}^{\tilde{\mu}_s} \lambda_k| \mathbb{1}_{(|\lambda_k - \mathbb{E}^{\tilde{\mu}_s} \lambda_k| \leq N^{-1 + \frac{\alpha_r}{2} + (t_r + 0.7P_r)\epsilon_0})} d\nu_s \\ &\leq \exp(-N^{0.19P_r\epsilon_0})(2|a| + 2|b|) + N^{-1 + \frac{\alpha_r}{2} + (t_r + 0.7P_r)\epsilon_0} \quad (\text{since } \lambda_k \in [a, b]) \\ &\leq 2N^{-1 + \frac{\alpha_r}{2} + (t_r + 0.7P_r)\epsilon_0}. \end{aligned}$$

So the first statement is proved. The second statement follows immediately. \square

17 Estimation of $|\gamma_k - \gamma_k^{(N)}|$ of scale $N^{-1 + \frac{\alpha_r}{2}}$

Recall that in Section 5 we defined $\{t_1, t_2, \dots\}$ and $\{P_1, P_2, \dots\}$ satisfying

1. $t_1 > P_1 > 0$.
2. For $k \in \{1, 2, \dots\}$, define $a_k = \frac{1}{2}(\frac{3}{4})^{k-1}$, $P_k = P_1 \times 0.2^{k-1}$ and $t_{k+1} = 2t_k + 1.6P_k$.

Suppose $r \in \{1, 2, \dots\}$. $\mathcal{L}(r)$ is defined (in Section 16) to be the following statement.

$\mathcal{L}(r)$: For any $\alpha \in (0, 1/2)$, there exist $N_0 = N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$ and $i \in [\alpha N, (1 - \alpha)N]$, then

$$\mathbb{P}^{\tilde{\mu}_s} \left(|\lambda_i - \gamma_i| > N^{-1 + a_r + t_r\epsilon_0} \right) \leq \exp(-N^{P_r\epsilon_0}).$$

We will use a similar method as in Section 14 to estimate $|\gamma_k - \gamma_k^{(N)}|$.

Lemma 17.1. Suppose Ω_L is a probability space and \mathbb{P}_L is the probability measure on it. Suppose $X : \Omega_L \rightarrow \mathbb{C}$ is a random variable. If $z_0 \in \mathbb{C}$, then $|\text{Var}_{\mathbb{P}_L}(X)| \leq \mathbb{E}^{\mathbb{P}_L}[|X - z_0|^2]$.

Proof. This lemma is trivial. \square

The following lemma is an analogue of the argument in the proof of Lemma 3.13 of [6].

Lemma 17.2. Suppose $P_r > 20$ (thus $t_r > 3$). Suppose $\mathcal{L}(r)$ is true and $\delta_* > 0$. Suppose

$$\Omega = \{E + i\eta | E \in [c + \delta_*, d - \delta_*], \eta \in [N^{-1+\frac{3}{4}a_r+(2t_r+P_r)\epsilon_0}, 1]\}.$$

There exist $N_0(V_p, \kappa, \epsilon_0, \delta_*, a_r, t_r, P_r) > 0$, $W_{10}(V_p, \kappa, \epsilon_0, \delta_*, a_r, t_r, P_r) > 0$ such that if $N > N_0$ and $z = E + i\eta \in \Omega$, then

$$\frac{1}{N^2} |\text{Var}_{\tilde{\mu}_s}(\sum \frac{1}{z - \lambda_k})| \leq \left(\frac{N^{\frac{3}{4}a_r}}{N\eta} + N^{-1+2.3\epsilon_0} + N^{-1+(2t_r+1.4P_r)\epsilon_0} \cdot \frac{1}{\eta} \right) W_{10}.$$

Proof. Suppose $\alpha = \alpha(\delta_*) \in (0, 1/2)$ such that $\alpha < \min(\int_c^{c+0.5\delta_*} \tilde{\rho}(t)dt, \int_{d-0.5\delta_*}^d \tilde{\rho}(t)dt)$. So

$$\gamma_i \in [c, c + 0.5\delta_*] \cup [d - 0.5\delta_*, d] \quad \forall i \in [1, \alpha N] \cup [(1 - \alpha)N, N]. \quad (17.74)$$

Suppose $i_0 \in \{1, \dots, N\}$ such that $|\gamma_{i_0}^{(N)} - E| \leq |\gamma_i^{(N)} - E|$ for all $i \in \{1, \dots, N\}$. In other words, $\gamma_{i_0}^{(N)}$ is closer to E than any other $\gamma_i^{(N)}$. So $i_0 \in [\alpha N, (1 - \alpha)N]$. Set

- $\text{Int} = \{i \in \mathbb{N} | |i - i_0| < N^{a_r+(t_r+0.7P_r)\epsilon_0}\},$
- $\text{Ext} = \{i \in [\alpha N, (1 - \alpha)N] \cap \mathbb{N} | |i - i_0| \geq N^{a_r+(t_r+0.7P_r)\epsilon_0}\},$
- $\text{Edg} = ([1, \alpha N] \cup ((1 - \alpha)N, N]) \cap \mathbb{N}.$

Set $\alpha_k = \mathbb{E}^{\tilde{\mu}_s} \lambda_k$. Suppose $z = E + i\eta \in \Omega$. According to Lemma 17.1,

$$\begin{aligned} & \frac{1}{N^2} |\text{Var}_{\tilde{\mu}_s}(\sum \frac{1}{z - \lambda_k})| \\ & \leq \frac{1}{N^2} \mathbb{E}^{\tilde{\mu}_s} \left[\left| \sum \frac{1}{z - \lambda_k} - \sum \frac{1}{z - \alpha_k} \right|^2 \right] \\ & \leq \frac{3}{N^2} \mathbb{E}^{\tilde{\mu}_s} \left[\left| \sum_{k \in \text{Edg}} \left(\frac{1}{z - \lambda_k} - \frac{1}{z - \alpha_k} \right) \right|^2 \right] + \frac{3}{N^2} \mathbb{E}^{\tilde{\mu}_s} \left[\left| \sum_{k \in \text{Ext}} \left(\frac{1}{z - \lambda_k} - \frac{1}{z - \alpha_k} \right) \right|^2 \right] \\ & \quad + \frac{3}{N^2} \mathbb{E}^{\tilde{\mu}_s} \left[\left| \sum_{k \in \text{Int}} \left(\frac{1}{z - \lambda_k} - \frac{1}{z - \alpha_k} \right) \right|^2 \right] \\ & := I + II + III. \end{aligned}$$

• **Estimation of I**

$$I \leq \frac{3}{N} \sum_{k \in \text{Edg}} \mathbb{E}^{\tilde{\mu}_s} \left[\left| \frac{1}{z - \lambda_k} - \frac{1}{z - \alpha_k} \right|^2 \right] = \frac{3}{N} \sum_{k \in \text{Edg}} \mathbb{E}^{\tilde{\mu}_s} \left[\left| \frac{\lambda_k - \alpha_k}{(z - \lambda_k)(z - \alpha_k)} \right|^2 \right]$$

According to Lemma 13.4, there is $N_0 = N_0(V_p, \kappa, \epsilon_0, \delta_*)$ such that when $N > N_0$ and $k \in [1, N]$,

$$\mathbb{P}^{\nu_s} \left(|E - \lambda_k| \geq \frac{\delta_*}{4}, |E - \alpha_k| < \frac{\delta_*}{5} \right) \leq \mathbb{P}^{\nu_s} \left(|\lambda_k - \alpha_k| > \frac{\delta_*}{20} \right) \leq \mathbb{P}^{\nu_s} \left(|\lambda_k - \alpha_k| > N^{-1/4} \right) \leq 2 \exp(-\beta \sqrt{N}).$$

So by Lemma 13.7,

$$\mathbb{P}^{\tilde{\mu}_s} \left(|E - \lambda_k| \geq \frac{\delta_*}{4}, |E - \alpha_k| < \frac{\delta_*}{5} \right) \leq \exp(-N^{0.24})$$

when $N > N_0(V_p, \kappa, \epsilon_0, \delta_*)$ and $k \in [1, N]$.

According to (17.74), if $N \geq 1$, $k \in \text{Edg}$ and $|E - \lambda_k| < \frac{\delta_*}{4}$, then $|\lambda_k - \gamma_k| > \frac{\delta_*}{4}$. So by Lemma 11.3, there exist $N_0 > 0$ depending on V_p, κ, ϵ_0 and δ_* such that when $N > N_0$,

$$\mathbb{P}^{\tilde{\mu}_s}(\exists k \in \text{Edg} \text{ st. } |E - \lambda_k| < \frac{\delta_*}{4}) \leq \exp(-N^{0.99}).$$

So for $N > N_0(V_p, \kappa, \epsilon_0, \delta_*)$ and $k \in \text{Edg}$,

$$\begin{aligned} & \mathbb{E}^{\tilde{\mu}_s} \left[\left| \frac{\lambda_k - \alpha_k}{(z - \lambda_k)(z - \alpha_k)} \right|^2 \right] \\ &= \int_{\tilde{\Sigma}_N} \left| \frac{\lambda_k - \alpha_k}{(z - \lambda_k)(z - \alpha_k)} \right|^2 \mathbb{1}_{(|E - \lambda_k| < \delta_*/4)} d\tilde{\mu}_s + \int_{\tilde{\Sigma}_N} \left| \frac{\lambda_k - \alpha_k}{(z - \lambda_k)(z - \alpha_k)} \right|^2 \mathbb{1}_{(|E - \lambda_k| \geq \delta_*/4, |E - \alpha_k| \geq \delta_*/5)} d\tilde{\mu}_s \\ &+ \int_{\tilde{\Sigma}_N} \left| \frac{\lambda_k - \alpha_k}{(z - \lambda_k)(z - \alpha_k)} \right|^2 \mathbb{1}_{(|E - \lambda_k| \geq \delta_*/4, |E - \alpha_k| < \delta_*/5)} d\tilde{\mu}_s \\ &\leq \frac{(2|a| + 2|b|)^2}{\eta^4} \mathbb{P}^{\tilde{\mu}_s}(|E - \lambda_k| < \delta_*/4) + \frac{400}{\delta_*^4} \mathbb{E}^{\tilde{\mu}_s}(|\alpha_k - \lambda_k|^2) \\ &+ \frac{16(2|a| + 2|b|)^2}{\eta^2 \delta_*^2} \mathbb{P}^{\tilde{\mu}_s}(|E - \lambda_k| \geq \delta_*/4, |E - \alpha_k| < \delta_*/5) \\ &\leq \frac{(2|a| + 2|b|)^2}{\eta^4} \exp(-N^{0.99}) + \frac{400}{\delta_*^4} \mathbb{E}^{\tilde{\mu}_s}(|\alpha_k - \lambda_k|^2) + \frac{16(2|a| + 2|b|)^2}{\eta^2 \delta_*^2} \cdot \exp(-N^{0.24}) \\ &\leq \exp(-N^{0.23}) + \frac{400}{\delta_*^4} \mathbb{E}^{\tilde{\mu}_s}(|\alpha_k - \lambda_k|^2) \quad (\text{since } \eta > N^{-1}). \end{aligned}$$

Therefore by Corollary 13.9, for $N > N_0(V_p, \kappa, \epsilon_0, \delta_*)$,

$$\begin{aligned} I &\leq \frac{3}{N} \sum_{k \in \text{Edg}} \left[\exp(-N^{0.23}) + \frac{400}{\delta_*^4} \mathbb{E}^{\tilde{\mu}_s}(|\alpha_k - \lambda_k|^2) \right] \\ &\leq \frac{3}{N} 2\alpha N \exp(-N^{0.23}) + \frac{3}{N} 2\alpha N \frac{400}{\delta_*^4} N^{-1+2.3\epsilon_0} \\ &\leq A_4 N^{-1+2.3\epsilon_0} \end{aligned}$$

for some $A_4 = A_4(V_p, \kappa, \epsilon_0, \delta_*) > 0$.

• Estimation of III

$$\begin{aligned} III &= \frac{3}{N^2} \mathbb{E}^{\tilde{\mu}_s} \left[\left| \sum_{k \in \text{Int}} \frac{\lambda_k - \alpha_k}{(z - \lambda_k)(z - \alpha_k)} \right|^2 \right] \leq \frac{3}{N^2 \eta^4} \mathbb{E}^{\tilde{\mu}_s} \left[\left| \sum_{k \in \text{Int}} (\lambda_k - \alpha_k) \right|^2 \right] \\ &\leq \frac{3}{N^2 \eta^4} \cdot 2N^{a_r + (t_r + 0.7P_r)\epsilon_0} \sum_{k \in \text{Int}} \mathbb{E}^{\tilde{\mu}_s} [(\lambda_k - \alpha_k)^2] \end{aligned}$$

When $N > N_0(V_p, \kappa, \epsilon_0, \delta_*, a_r, t_r, P_r)$, we have $\text{Int} \subset [\alpha N, (1 - \alpha)N]$, thus by Theorem 16.10, for each $k \in \text{Int}$,

$$\mathbb{P}^{\tilde{\mu}_s} \left(\left| \lambda_k - \alpha_k \right| > N^{-1 + \frac{a_r}{2} + (t_r + 0.7P_r)\epsilon_0} \right) \leq \exp(-N^{0.2P_r\epsilon_0}).$$

Then for $N > N_0$ and $k \in \text{Int}$,

$$\begin{aligned}\mathbb{E}^{\tilde{\mu}_s} \left[(\lambda_k - \alpha_k)^2 \right] &= \int_{\tilde{\Sigma}_N} (\lambda_k - \alpha_k)^2 \mathbb{1}_{(|\lambda_k - \alpha_k| > N^{-1 + \frac{a_r}{2} + (t_r + 0.7P_r)\epsilon_0})} d\tilde{\mu}_s + \int_{\tilde{\Sigma}_N} (\lambda_k - \alpha_k)^2 \mathbb{1}_{(|\lambda_k - \alpha_k| \leq N^{-1 + \frac{a_r}{2} + (t_r + 0.7P_r)\epsilon_0})} d\tilde{\mu}_s \\ &\leq (2|a| + 2|b|)^2 \exp(-N^{0.2P_r\epsilon_0}) + \left(N^{-1 + \frac{a_r}{2} + (t_r + 0.7P_r)\epsilon_0} \right)^2 \\ &\leq 2N^{-2 + a_r + (1.4P_r + 2t_r)\epsilon_0}\end{aligned}$$

and

$$III \leq \frac{3}{N^2\eta^4} \cdot 2N^{a_r + (t_r + 0.7P_r)\epsilon_0} \cdot 2N^{a_r + (t_r + 0.7P_r)\epsilon_0} \cdot 2N^{-2 + a_r + (2t_r + 1.4P_r)\epsilon_0} \leq 24 \frac{N^{\frac{3}{4}a_r}}{N\eta}.$$

The last step comes from the fact that $\eta \geq N^{-1 + \frac{3}{4}a_r + (2t_r + P_r)\epsilon_0}$.

• Estimation of II

Set $r_m = \min_{x \in [c + \frac{\delta_*}{2}, d - \frac{\delta_*}{2}]} \rho(x) > 0$. So when $N > N_0(V_p, \kappa, \epsilon_o, \delta_*)$, $|\gamma_{i_0} - E| \leq \frac{1}{Nr_m}$ and $|\gamma_{i_0} - \gamma_k| \geq \|\tilde{\rho}\|_\infty^{-1} N^{-1 + a_r + (t_r + 0.7P_r)\epsilon_0}$ ($k \in \text{Ext}$).
Set

$$A_k = \{\lambda \in \tilde{\Sigma}_N \mid |\lambda_k - \alpha_k| \leq 3N^{-1 + \frac{a_r}{2} + (t_r + 0.7P_r)\epsilon_0}, |\lambda_k - \gamma_k| \leq N^{-1 + a_r + t_r\epsilon_0}\}.$$

Suppose $k \in \text{Ext}$ and $\lambda \in A_k$, then we have the following estimations:

- Notice that $E - \lambda_k = (\gamma_{i_0} - \gamma_k) + (E - \gamma_{i_0}) + (\gamma_k - \lambda_k)$ where $|\gamma_{i_0} - \gamma_k| \geq \|\tilde{\rho}\|_\infty^{-1} N^{-1 + a_r + (t_r + 0.7P_r)\epsilon_0}$, $|\gamma_k - \lambda_k| \leq N^{-1 + a_r + t_r\epsilon_0}$ and $|E - \gamma_{i_0}| \leq \frac{1}{Nr_m}$ when $N > N_0(V_p, \kappa, \epsilon_o, \delta_*)$. So when $N > N_0(V_p, \kappa, \epsilon_o, \delta_*, a_r, t_r, P_r)$, we have $\frac{1}{2}|E - \lambda_k| \leq |\gamma_{i_0} - \gamma_k| \leq 2|E - \lambda_k|$.
- Notice that $E - \alpha_k = (\gamma_{i_0} - \gamma_k) + (E - \gamma_{i_0}) + (\gamma_k - \lambda_k) + (\lambda_k - \alpha_k)$ where $|\gamma_{i_0} - \gamma_k| \geq \|\tilde{\rho}\|_\infty^{-1} N^{-1 + a_r + (t_r + 0.7P_r)\epsilon_0}$, $|\gamma_k - \lambda_k| \leq N^{-1 + a_r + t_r\epsilon_0}$, $|\lambda_k - \alpha_k| \leq 3N^{-1 + \frac{a_r}{2} + (t_r + 0.7P_r)\epsilon_0}$ and $|E - \gamma_{i_0}| \leq \frac{1}{Nr_m}$ when $N > N_0(V_p, \kappa, \epsilon_o, \delta_*)$. So when $N > N_0(V_p, \kappa, \epsilon_o, \delta_*, a_r, t_r, P_r)$, we have $\frac{1}{2}|E - \alpha_k| \leq |\gamma_{i_0} - \gamma_k| \leq 2|E - \alpha_k|$.

So by the above estimations, when $k \in \text{Ext}$, $\lambda \in A_k$ and $N > N_0(V_p, \kappa, \epsilon_o, \delta_*, a_r, t_r, P_r)$,

$$|z - \lambda_k| |z - \alpha_k| = \sqrt{\eta^2 + (E - \lambda_k)^2} \sqrt{\eta^2 + (E - \alpha_k)^2} \geq \eta^2 + \frac{1}{4} |\gamma_{i_0} - \gamma_k|^2.$$

Set $A = \bigcap_{k \in \text{Ext}} A_k$. According to Theorem 16.10 and $\mathcal{L}(r)$, there exist $N_0 = N_0(V_p, \kappa, \epsilon_o, \delta_*, a_r, t_r, P_r) > 0$ such that if $N > N_0$, then

$$\mathbb{P}^{\tilde{\mu}_s}(A_k^c) \leq 2 \exp(-N^{0.2P_r\epsilon_0}), \quad \forall k \in \text{Ext}$$

and thus

$$\mathbb{P}^{\tilde{\mu}_s}(A^c) \leq \exp(-N^{0.18P_r\epsilon_0}).$$

Notice that if $N \geq 1$ and $j_1, j_2 \in [1, N]$, then

$$|\gamma_{j_1} - \gamma_{j_2}| \|\tilde{\rho}\|_\infty \geq \int_{\gamma_{j_1}}^{\gamma_{j_2}} \tilde{\rho}(t) dt = \frac{|j_1 - j_2|}{N}$$

and therefore

$$|\gamma_{j_1} - \gamma_{j_2}| \geq \|\tilde{\rho}\|_\infty^{-1} \frac{|i - j|}{N}.$$

When $N > N_0(V_p, \kappa, \epsilon_0, a_r, \delta_*, t_r, P_r)$,

$$\begin{aligned} II &= \frac{3}{N^2} \mathbb{E}^{\tilde{\mu}_s} \left[\left| \sum_{k \in \text{Ext}} \left(\frac{1}{z - \lambda_k} - \frac{1}{z - \alpha_k} \right) \right|^2 \right] \leq \frac{3}{N^2} \mathbb{E}^{\tilde{\mu}_s} \left[\left(\sum_{k \in \text{Ext}} \frac{|\alpha_k - \lambda_k|}{|z - \alpha_k| |z - \lambda_k|} \right)^2 \right] \\ &= \frac{3}{N^2} \int_A \left(\sum_{k \in \text{Ext}} \frac{|\alpha_k - \lambda_k|}{|z - \alpha_k| |z - \lambda_k|} \right)^2 d\tilde{\mu}_s + \frac{3}{N^2} \int_{A^c} \left(\sum_{k \in \text{Ext}} \frac{|\alpha_k - \lambda_k|}{|z - \alpha_k| |z - \lambda_k|} \right)^2 d\tilde{\mu}_s \\ &\leq \frac{3}{N^2} \left[\sum_{k \in \text{Ext}} \frac{3N^{-1+\frac{a_r}{2}+(t_r+0.7P_r)\epsilon_0}}{\eta^2 + \frac{1}{4}|\gamma_{i_0} - \gamma_k|^2} \right]^2 + \frac{3}{N^2} \left[\sum_{k \in \text{Ext}} \frac{2|a| + 2|b|}{\eta^2} \right]^2 \mathbb{P}^{\tilde{\mu}_s}(A^c) \\ &\leq 27N^{-4+a_r+(2t_r+1.4P_r)\epsilon_0} \left[\sum_{k \in \text{Ext}} \frac{1}{\eta^2 + \frac{1}{4}|\gamma_{i_0} - \gamma_k|^2} \right]^2 + \frac{12(|a| + |b|)^2}{\eta^4} \exp(-N^{0.18P_r}\epsilon_0) \\ &\leq 27 \max(1, 16\|\tilde{\rho}\|_\infty^4) N^{-4+a_r+(2t_r+1.4P_r)\epsilon_0} \left[\sum_{k \in \text{Ext}} \frac{1}{\eta^2 + \frac{|k-i_0|^2}{N^2}} \right]^2 + 12(|a| + |b|)^2 N^4 \exp(-N^{0.18P_r}\epsilon_0) \end{aligned}$$

Set $E_1 = \mathbb{N} \cap [i_0 + N^{a_r+(t_r+0.7P_r)\epsilon_0}, (1-\alpha)N]$ and $E_2 = \mathbb{N} \cap [\alpha N, i_0 - N^{a_r+(t_r+0.7P_r)\epsilon_0}]$. Then $\text{Ext} = E_1 \cup E_2$.

- If $\eta \leq N^{-1+a_r}$, then

$$\sum_{k \in E_1} \frac{1}{\eta^2 + \frac{|k-i_0|^2}{N^2}} \leq \sum_{k=i_0+N^{a_r+(t_r+0.7P_r)\epsilon_0}}^{(1-\alpha)N} \frac{N^2}{(k-i_0)^2} \leq 2N^{2-a_r-(t_r+0.7P_r)\epsilon_0} \leq 2 \frac{N^{\frac{3}{2}-\frac{1}{2}a_r-(t_r+0.7P_r)\epsilon_0}}{\sqrt{\eta}}.$$

- If $N^{-1+a_r} \leq \eta \leq \frac{i_0}{N}$, then

$$\begin{aligned} \sum_{k \in E_1} \frac{1}{\eta^2 + \frac{|k-i_0|^2}{N^2}} &\leq \sum_{N^{a_r+(t_r+0.7P_r)\epsilon_0} \leq k-i_0 \leq \eta N} \frac{1}{\eta^2} + \sum_{k=i_0+\eta N}^{(1-\alpha)N} \frac{N^2}{(k-i_0)^2} \leq \frac{N\eta}{\eta^2} + N^2 \frac{1}{\eta N - 1} \\ &\leq \frac{3N}{\eta} \leq 3 \frac{N^{\frac{3}{2}-\frac{1}{2}a_r}}{\sqrt{\eta}}. \end{aligned}$$

- If $\frac{i_0}{N} \leq \eta \leq 1$, then $\sum_{k \in E_1} \frac{1}{\eta^2 + \frac{|k-i_0|^2}{N^2}} \leq \sum_{k \in E_1} \frac{1}{\eta^2} \leq \frac{N}{\eta^2} \leq \alpha^{-\frac{3}{2}} \frac{N}{\sqrt{\eta}}$ since $i_0 \geq \alpha N$.

We can estimate $\sum_{k \in E_2} \frac{1}{\eta^2 + \frac{|k-i_0|^2}{N^2}}$ in exactly the same way. So for $N > N_0(V_p, \kappa, \epsilon_0, a_r, \delta_*, t_r, P_r)$ and $z \in \Omega$,

$$\sum_{k \in \text{Ext}} \frac{1}{\eta^2 + \frac{|k-i_0|^2}{N^2}} = \sum_{k \in E_1} \frac{1}{\eta^2 + \frac{|k-i_0|^2}{N^2}} + \sum_{k \in E_2} \frac{1}{\eta^2 + \frac{|k-i_0|^2}{N^2}} \leq 2(\alpha^{-3/2} + 3) \frac{N^{\frac{3}{2}-\frac{1}{2}a_r}}{\sqrt{\eta}}$$

and

$$\begin{aligned} II &\leq 27 \max(1, 16 \|\tilde{\rho}\|_\infty^4) N^{-4+a_r+(2t_r+1.4P_r)\epsilon_0} \left[2(\alpha^{-3/2} + 3) \frac{N^{\frac{3}{2}-\frac{1}{2}a_r}}{\sqrt{\eta}} \right]^2 + 12(|a| + |b|)^2 N^4 \exp(-N^{0.18P_r\epsilon_0}) \\ &\leq T_{16} N^{-1+(2t_r+1.4P_r)\epsilon_0} \cdot \frac{1}{\eta} \end{aligned}$$

where T_{16} depends on $V_p, \kappa, \epsilon_0, \delta_*, a_r, t_r, P_r$.

By the estimations of I, II and III , when $N > N_0(V_p, \kappa, \epsilon_0, a_r, t_r, \delta_*, P_r)$ and $z \in \Omega$,

$$\frac{1}{N^2} |\text{Var}_{\nu_s} \left(\sum \frac{1}{z - \lambda_k} \right)| \leq 24 \frac{N^{\frac{3}{4}a_r}}{N\eta} + T_{16} N^{-1+(2t_r+1.4P_r)\epsilon_0} \cdot \frac{1}{\eta} + A_4 N^{-1+2.3\epsilon_0}.$$

□

Suppose $\chi(x)$ is a smooth even function with $\chi(x) = 0$ on $[-10\kappa, 10\kappa]^c$, $\chi(x) = 1$ on $[-5\kappa, 5\kappa]$ and $\|\chi\|_\infty = 1$.

Suppose $\psi(x)$ is a smooth function satisfying the following conditions.

1. $\psi(x) = 1$ if $x \leq 0$.
2. $\psi(x) = 0$ if $x \geq 1$
3. $\psi(x) \in [0, 1]$ for all $x \in \mathbb{R}$.

Set $\eta_1 = \eta_1(N) = N^{-1+\frac{3}{4}a_r+(2t_r+P_r)\epsilon_0}$.

For $E \in [c - \frac{\kappa}{4}, d + \frac{\kappa}{4}]$, set

$$f_E(x) = \begin{cases} \psi(\frac{x-E}{2\eta_1} + \frac{1}{2}) & \text{if } x \geq a \\ 1 & \text{if } a - N^{\epsilon_0} \leq x \leq a \\ \psi(a - N^{\epsilon_0} - x) & \text{if } x \leq a - N^{\epsilon_0} \end{cases}$$

So

$$f_E(x) = \begin{cases} 0 & \text{if } x \in (-\infty, a - N^{\epsilon_0} - 1] \cup [E + \eta_1, +\infty) \\ 1 & \text{if } x \in [a - N^{\epsilon_0}, E - \eta_1]. \end{cases}$$

Moreover, if $a - N^{\epsilon_0} - 1 \leq x \leq a - N^{\epsilon_0}$, then $|f'_E(x)| \leq \|\psi'\|_\infty$ and $|f''_E(x)| \leq \|\psi''\|_\infty$; if $E - \eta_1 \leq x \leq E + \eta_1$, then $|f'_E(x)| \leq \frac{1}{2\eta_1} \|\psi'\|_\infty$ and $|f''_E(x)| \leq \frac{1}{4\eta_1^2} \|\psi''\|_\infty$. Suppose $\bar{\rho}(t) = \tilde{\rho}(t) - \tilde{\rho}_1^{(N)}(t)$ and $\bar{m}(z) = \int_a^b \frac{1}{z-t} \bar{\rho}(t) dt$.

Lemma 17.3. *Suppose $P_r > 20$ (thus $t_r > 3$). Suppose $\mathcal{L}(r)$ is true and $\delta_* > 0$. Suppose $0.4P_r\epsilon_0 < \frac{3}{4}a_r$.*

There exist $M_6 = M_6(V_p, \kappa, \epsilon_0, \delta_, a_r, t_r, P_r) > 0$ and $N_0 = N_0(V_p, \kappa, \epsilon_0, \delta_*, a_r, t_r, P_r) > 0$ such that if $N > N_0$ and $E \in [c + 2\delta_*, d - 2\delta_*]$, then*

$$\left| \int f_E(t) \bar{\rho}(t) dt \right| \leq M_6 \cdot N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0}.$$

Proof. By Lemma 14.1 we have:

$$\begin{aligned}
& \left| \int f_E(t) \bar{\rho}(t) dt \right| \\
& \leq \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} y f_E''(x) \chi(y) \operatorname{Im} \bar{m}(x + iy) dx dy \right| \\
& + \frac{1}{2\pi} \int_{\mathbb{R}^2} |f_E(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy + \frac{1}{2\pi} \int_{\mathbb{R}^2} |y| |f_E'(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy \\
& = \frac{1}{\pi} \left| \int_0^{10\kappa} \int_{a-N^{\epsilon_0}-1}^{a-N^{\epsilon_0}} y f_E''(x) \chi(y) \operatorname{Im} \bar{m}(x + iy) dx dy \right| + \frac{1}{\pi} \left| \int_0^{10\kappa} \int_{E-\eta_1}^{E+\eta_1} y f_E''(x) \chi(y) \operatorname{Im} \bar{m}(x + iy) dx dy \right| \\
& + \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{a-N^{\epsilon_0}-1}^{E+\eta_1} |f_E(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy + \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{a-N^{\epsilon_0}-1}^{a-N^{\epsilon_0}} |y| |f_E'(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy \\
& + \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{E-\eta_1}^{E+\eta_1} |y| |f_E'(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy \\
& := I + II + III + IV + V.
\end{aligned}$$

• **Estimation of III, IV and V**

Fix $k_0 = k_0(\epsilon_0) \in \mathbb{N}$ such that $a_{k_0} := \left(\frac{1}{2}\right)^{k_0} < \frac{1}{4}\epsilon_0$. According to Theorem 9.6 with $h \equiv 0$, there exist $C_{ind} = C_{ind}(V_p, \kappa, \epsilon_0) > 0$ and $N_0 = N_0(V_p, \kappa, \epsilon_0) > 0$ such that for any $N > N_0$ and $\operatorname{Im} z \in [N^{-a_{k_0}}, 10\kappa)$,

$$|(z - c)(z - d)|^{1/2} |\tilde{m}_N(z) - \tilde{m}(z)| \leq N^{-1+\frac{3}{2}\epsilon_0} \ln N \cdot C_{ind}.$$

If $N > N_1(\kappa, \epsilon_0)$, then $\ln N < N^{\epsilon_0/2}$ and $[5\kappa, 10\kappa) \subset [N^{-a_{k_0}}, 10\kappa)$ thus $|(z - c)(z - d)|^{1/2} \geq 5\kappa$ for $z \in \{z | \operatorname{Im} z \geq 5\kappa\}$. Therefore when $N > N_0(V_p, \kappa, \epsilon_0)$ we have $|\tilde{m}_N(z) - \tilde{m}(z)| \leq \frac{1}{5\kappa} N^{-1+2\epsilon_0} C_{ind}$ for $\operatorname{Im} z \in (5\kappa, 10\kappa)$ and

$$\begin{aligned}
III &= \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{a-N^{\epsilon_0}-1}^{E+\eta_1} |f_E(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy \\
&\leq \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{a-N^{\epsilon_0}-1}^{E+\eta_1} |f_E(x)| |\chi'(y)| \left(N^{-1+2\epsilon_0} \frac{1}{5\kappa} C_{ind} \right) dx dy \\
&\leq \frac{1}{\pi} C_{ind} \|\chi'\|_{\infty} (d - c + \kappa + 2 + N^{\epsilon_0}) N^{-1+2\epsilon_0} \\
&\leq \frac{2}{\pi} C_{ind} \|\chi'\|_{\infty} N^{-1+3\epsilon_0},
\end{aligned}$$

$$\begin{aligned}
IV &= \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{a-N^{\epsilon_0}-1}^{a-N^{\epsilon_0}} |y| |f_E'(x)| |\chi'(y)| |\bar{m}(x + iy)| dx dy \\
&\leq \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{a-N^{\epsilon_0}-1}^{a-N^{\epsilon_0}} |y| |f_E'(x)| |\chi'(y)| \left(N^{-1+2\epsilon_0} \frac{1}{5\kappa} C_{ind} \right) dx dy \\
&\leq \frac{10\kappa}{\pi} C_{ind} \|\psi'\|_{\infty} \|\chi'\|_{\infty} N^{-1+2\epsilon_0},
\end{aligned}$$

$$\begin{aligned}
V &= \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{E-\eta_1}^{E+\eta_1} |y| |f'_E(x)| |\chi'(y)| |\bar{m}(x+iy)| dx dy \\
&\leq \frac{1}{\pi} \int_{5\kappa}^{10\kappa} \int_{E-\eta_1}^{E+\eta_1} |y| |f'_E(x)| |\chi'(y)| \left(N^{-1+2\epsilon_0} \frac{1}{5\kappa} C_{ind} \right) dx dy \\
&\leq \frac{10\kappa}{\pi} C_{ind} \|\psi'\|_\infty \|\chi'\|_\infty N^{-1+2\epsilon_0}.
\end{aligned}$$

• **Estimation of I**

Set $y_0 = N^{-1}$. For $x \in \mathbb{R}$ and $y \in (0, N^{-1})$, $y |\operatorname{Im} \tilde{m}_N(x+iy)| \leq y_0 |\operatorname{Im} \tilde{m}_N(x+iy_0)|$. Then,

$$\begin{aligned}
y |\operatorname{Im} \tilde{m}(x+iy)| &= y |\operatorname{Im}(\tilde{m}_N(x+iy) - \tilde{m}(x+iy))| \leq y |\operatorname{Im}(\tilde{m}_N(x+iy))| + y\pi \|\tilde{\rho}\|_\infty \quad (\text{see Lemma 10.2}). \\
&\leq y_0 |\operatorname{Im}(\tilde{m}_N(x+iy_0))| + y_0\pi \|\tilde{\rho}\|_\infty \leq y_0 |\operatorname{Im}(\tilde{m}_N(x+iy_0) - \tilde{m}(x+iy_0))| + 2y_0\pi \|\tilde{\rho}\|_\infty \\
&\leq y_0 |\tilde{m}(x+iy_0)| + 2y_0\pi \|\tilde{\rho}\|_\infty.
\end{aligned}$$

According to Corollary 10.5, there are $N_0 = N_0(\epsilon_0, V_p, \kappa) > 0$, $W_8 = W_8(V_p, \kappa, \epsilon_0) > 0$ such that if $N > N_0$, $z = E + i\eta$ with $E \in [a - N^{\epsilon_0} - 1, a - N^{\epsilon_0}]$ and $\eta > N^{-1}$, then $|\tilde{m}_N(z) - \tilde{m}(z)| \leq W_8 N^{-1+\epsilon_0}$. Therefore when $N > N_0(\epsilon_0, V_p, \kappa) > 0$,

$$\begin{aligned}
I &= \frac{1}{\pi} \left| \int_0^{10\kappa} \int_{a-N^{\epsilon_0}-1}^{a-N^{\epsilon_0}} y f''_E(x) \chi(y) \operatorname{Im} \tilde{m}(x+iy) dx dy \right| \\
&\leq \frac{1}{\pi} \left| \int_0^{N^{-1}} \int_{a-N^{\epsilon_0}-1}^{a-N^{\epsilon_0}} y f''_E(x) \chi(y) \operatorname{Im} \tilde{m}(x+iy) dx dy \right| + \frac{1}{\pi} \left| \int_{N^{-1}}^{10\kappa} \int_{a-N^{\epsilon_0}-1}^{a-N^{\epsilon_0}} y f''_E(x) \chi(y) \operatorname{Im} \tilde{m}(x+iy) dx dy \right| \\
&\leq \frac{1}{\pi} \int_0^{N^{-1}} \int_{a-N^{\epsilon_0}-1}^{a-N^{\epsilon_0}} |f''_E(x) \chi(y)| (y_0 W_8 N^{-1+\epsilon_0} + 2y_0\pi \|\tilde{\rho}\|_\infty) dx dy \quad (\text{since } |\tilde{m}(x+iy_0)| \leq W_8 N^{-1+\epsilon_0}) \\
&\quad + \frac{1}{\pi} \left| \int_{N^{-1}}^{10\kappa} \int_{a-N^{\epsilon_0}-1}^{a-N^{\epsilon_0}} y f''_E(x) \chi(y) W_8 N^{-1+\epsilon_0} dx dy \right| \\
&\leq \frac{1}{\pi} \frac{1}{N} \|\psi''\|_\infty (W_8 N^{-2+\epsilon_0} + 2\pi N^{-1} \|\tilde{\rho}\|_\infty) + \frac{1}{\pi} 100\kappa^2 \|\psi''\|_\infty W_8 N^{-1+\epsilon_0} \\
&\leq W_9 N^{-1+\epsilon_0}
\end{aligned}$$

where $W_9 = W_9(V_p, \kappa, \epsilon_0) > 0$.

• **Estimation of II**

When $N > N_0(V_p, \kappa, \epsilon_0, a_r, \delta_*)$, $[E - \eta_1, E + \eta_1] \subset [c + \delta_*, d - \delta_*]$.

Set $y_0 = \eta_1 = N^{-1+\frac{3}{4}a_r+(2t_r+P_r)\epsilon_0}$.

By to Lemma 17.2 and Theorem 10.4 (with the assumption $0.4P_r\epsilon_0 < \frac{3}{4}a_r$), there exist $M_3 = M_3(V_p, \kappa, \epsilon_0, \delta_*, a_r, t_r, P_r) > 0$ and $N_0 = N_0(V_p, \kappa, \epsilon_0, \delta_*, a_r, t_r, P_r) > 0$ such that if $N > N_0$, $x \in [E - \eta_1, E + \eta_1]$ and $y \in [y_0, 3.5\kappa]$, then

$$|\bar{m}(x+iy)| \leq M_3 \left(\frac{N^{\frac{3}{4}a_r}}{N\eta} + \frac{N^{\epsilon_0}}{N\eta} + N^{-1+2.3\epsilon_0} + N^{-1+(2t_r+1.4P_r)\epsilon_0} \cdot \frac{1}{\eta} \right) \leq M_3 \frac{1}{N\eta} N^{\frac{3}{4}a_r+(2t_r+1.4P_r)\epsilon_0}. \quad (17.75)$$

To estimate II , notice that $II \leq VI + VII + VIII$ where

$$\begin{aligned} VI &= \frac{1}{\pi} \left| \int_0^{y_0} \int_{E-\eta_1}^{E+\eta_1} y f_E''(x) \chi(y) \operatorname{Im} \tilde{m}(x + iy) dx dy \right|, \\ VII &= \frac{1}{\pi} \left| \int_{y_0}^{3.5\kappa} \int_{E-\eta_1}^{E+\eta_1} y f_E''(x) \chi(y) \operatorname{Im} \tilde{m}(x + iy) dx dy \right|, \\ VIII &= \frac{1}{\pi} \left| \int_{3.5\kappa}^{10\kappa} \int_{E-\eta_1}^{E+\eta_1} y f_E''(x) \chi(y) \operatorname{Im} \tilde{m}(x + iy) dx dy \right|. \end{aligned}$$

• **Estimation of VI**

Suppose $x \in [E - \eta_1, E + \eta_1]$ and $y \in (0, y_0]$. By direct computation, $y |\operatorname{Im} \tilde{m}_N(x + iy)| \leq y_0 |\operatorname{Im} \tilde{m}_N(x + iy_0)|$. Then,

$$\begin{aligned} y |\operatorname{Im} \tilde{m}(x + iy)| &= y |\operatorname{Im}(\tilde{m}_N(x + iy) - \tilde{m}(x + iy))| \leq y |\operatorname{Im}(\tilde{m}_N(x + iy))| + y \pi \|\tilde{\rho}\|_\infty \quad (\text{see Lemma 10.2}). \\ &\leq y_0 |\operatorname{Im}(\tilde{m}_N(x + iy_0))| + y_0 \pi \|\tilde{\rho}\|_\infty \leq y_0 |\operatorname{Im}(\tilde{m}_N(x + iy_0) - \tilde{m}(x + iy_0))| + 2y_0 \pi \|\tilde{\rho}\|_\infty \\ &\leq y_0 |\operatorname{Im}(\tilde{m}_N(x + iy_0) - \tilde{m}(x + iy_0))| + 2y_0 \pi \|\tilde{\rho}\|_\infty. \end{aligned}$$

By (17.75), when $N > N_0(V_p, \kappa, \epsilon_0, \delta_*, a_r, t_r, P_r)$, we have

$$\begin{aligned} VI &= \frac{1}{\pi} \left| \int_0^{y_0} \int_{E-\eta_1}^{E+\eta_1} y f_E''(x) \chi(y) \operatorname{Im} \tilde{m}(x + iy) dx dy \right| \\ &\leq \frac{1}{\pi} \left| \int_0^{y_0} \int_{E-\eta_1}^{E+\eta_1} f_E''(x) \chi(y) \left(y_0 \operatorname{Im} \tilde{m}(x + iy_0) + 2y_0 \pi \|\tilde{\rho}\|_\infty \right) dx dy \right| \\ &\leq \frac{1}{\pi} \left| \int_0^{y_0} \int_{E-\eta_1}^{E+\eta_1} f_E''(x) \chi(y) \left(M_3 \frac{N^{\frac{3}{4}a_r + (2t_r + 1.4P_r)\epsilon_0}}{N} + 2y_0 \pi \|\tilde{\rho}\|_\infty \right) dx dy \right| \\ &\leq \frac{\|\psi''\|_\infty}{2\pi} \left[M_3 N^{-1 + \frac{3}{4}a_r + (2t_r + 1.4P_r)\epsilon_0} + 2\pi \|\tilde{\rho}\|_\infty N^{-1 + \frac{3}{4}a_r + (2t_r + P_r)\epsilon_0} \right] \quad (\text{since } y_0 = \eta_1 = N^{-1 + \frac{3}{4}a_r + (2t_r + P_r)\epsilon_0}) \\ &\leq M_3 \|\psi''\|_\infty N^{-1 + \frac{3}{4}a_r + (2t_r + 1.4P_r)\epsilon_0}. \end{aligned}$$

• **Estimation of VII**

To estimate VII , notice that $\tilde{m}(z)$ is analytic on a neighborhood of the domain of the integral: $\{x + iy | x \in (E - \eta_1, E + \eta_1), y \in (y_0, 4\kappa)\}$. So on this domain $\frac{\partial}{\partial x} \operatorname{Im} \tilde{m}(x + iy) = -\frac{\partial}{\partial y} \operatorname{Re} \tilde{m}(x + iy)$.

Thus when $N > N_0(V_p, \kappa, \epsilon_0, \delta_*, a_r, t_r, P_r)$,

$$\begin{aligned}
VII &= \frac{1}{\pi} \left| \int_{y_0}^{3.5\kappa} \int_{E-\eta_1}^{E+\eta_1} y f'_E(x) \chi(y) \frac{\partial}{\partial x} (\text{Im} \bar{m}(x + iy)) dx dy \right| \quad (\text{integral by parts}) \\
&= \frac{1}{\pi} \left| \int_{y_0}^{3.5\kappa} \int_{E-\eta_1}^{E+\eta_1} y f'_E(x) \chi(y) \frac{\partial}{\partial y} (\text{Re} \bar{m}(x + iy)) dx dy \right| \\
&\leq \frac{1}{\pi} \left| \int_{E-\eta_1}^{E+\eta_1} (\text{Re} \bar{m}(x + i3.5\kappa)) 3.5\kappa \chi(3.5\kappa) f'_E(x) dx \right| + \frac{1}{\pi} \left| \int_{E-\eta_1}^{E+\eta_1} (\text{Re} \bar{m}(x + iy_0)) y_0 \chi(y_0) f'_E(x) dx \right| \\
&\quad + \frac{1}{\pi} \left| \int_{E-\eta_1}^{E+\eta_1} \int_{y_0}^{3.5\kappa} (\text{Re} \bar{m}(x + iy)) \frac{\partial}{\partial y} (y \chi(y)) f'_E(x) dy dx \right| \\
&\leq \frac{1}{\pi} 2\eta_1 \left(M_3 \frac{N^{\frac{3}{4}a_r + (2t_r + 1.4P_r)\epsilon_0}}{3.5\kappa N} \right) 3.5\kappa \frac{\|\psi'\|_\infty}{2\eta_1} + \frac{1}{\pi} 2\eta_1 \left(M_3 \frac{N^{\frac{3}{4}a_r + (2t_r + 1.4P_r)\epsilon_0}}{Ny_0} \right) y_0 \frac{\|\psi'\|_\infty}{2\eta_1} \\
&\quad + \frac{1}{\pi} \left| \int_{E-\eta_1}^{E+\eta_1} \int_{y_0}^{3.5\kappa} \left(M_3 \frac{N^{\frac{3}{4}a_r + (2t_r + 1.4P_r)\epsilon_0}}{Ny} \right) (\chi(y) + y \chi'(y)) \frac{\|\psi'\|_\infty}{2\eta_1} dy dx \right| \\
&\leq \frac{2}{\pi} M_3 \|\psi'\|_\infty N^{-1 + \frac{3}{4}a_r + (2t_r + 1.4P_r)\epsilon_0} + \frac{1}{\pi} M_3 (1 + \|\chi'\|_\infty) \|\psi'\|_\infty N^{-1 + \frac{3}{4}a_r + (2t_r + 1.4P_r)\epsilon_0} \int_{y_0}^{3.5\kappa} \frac{1}{y} dy \\
&\leq M_4 N^{-1 + \frac{3}{4}a_r + (2t_r + 1.4P_r)\epsilon_0} \ln N \\
&\leq M_4 N^{-1 + \frac{3}{4}a_r + (2t_r + 1.5P_r)\epsilon_0}
\end{aligned}$$

for some $M_4 = M_4(V_p, \kappa, \epsilon_0, \delta_*, a_r, t_r, P_r) > 0$.

• **Estimation of VIII**

Fix $k_1 = k_1(\epsilon_0) \in \mathbb{N}$ such that $a_{k_1} := \left(\frac{1}{2}\right)^{k_1} < \frac{1}{4}\epsilon_0$. According to Theorem 9.6 with $h \equiv 0$, there exist $C_{ind} = C_{ind}(V_p, \kappa, \epsilon_0) > 0$ and $N_0 = N_0(V_p, \kappa, \epsilon_0) > 0$ such that for any $N > N_0$ and $\text{Im} z \in [N^{-a_{k_1}}, 10\kappa)$,

$$|(z - c)(z - d)|^{1/2} |\tilde{m}_N(z) - \tilde{m}(z)| \leq N^{-1 + \frac{3}{2}\epsilon_0} \ln N \cdot C_{ind}.$$

If $N > N_1(\kappa, \epsilon_0)$, then $\ln N < N^{\epsilon_0/2}$ and $[3.5\kappa, 10\kappa) \subset [N^{-a_{k_1}}, 10\kappa)$ thus $|(z - c)(z - d)|^{1/2} \geq 3.5\kappa$ for $z \in \{z | \text{Im} z \geq 3.5\kappa\}$. Therefore when $N > N_0(V_p, \kappa, \epsilon_0)$ we have $|\tilde{m}_N(z) - \tilde{m}(z)| \leq \frac{1}{3.5\kappa} N^{-1 + 2\epsilon_0} C_{ind}$ for $\text{Im} z \in (3.5\kappa, 10\kappa)$ and

$$\begin{aligned}
VIII &= \frac{1}{\pi} \left| \int_{3.5\kappa}^{10\kappa} \int_{E-\eta_1}^{E+\eta_1} y f_E''(x) \chi(y) \operatorname{Im} \bar{m}(x+iy) dx dy \right| \\
&= \frac{1}{\pi} \left| \int_{3.5\kappa}^{10\kappa} \int_{E-\eta_1}^{E+\eta_1} y f_E'(x) \chi(y) \frac{\partial}{\partial x} (\operatorname{Im} \bar{m}(x+iy)) dx dy \right| \quad (\text{integral by parts}) \\
&= \frac{1}{\pi} \left| \int_{3.5\kappa}^{10\kappa} \int_{E-\eta_1}^{E+\eta_1} y f_E'(x) \chi(y) \frac{\partial}{\partial y} (\operatorname{Re} \bar{m}(x+iy)) dx dy \right| \quad (\text{since } \bar{m} \text{ is analytic on a neighbor of the domain}) \\
&\leq \frac{1}{\pi} \left| \int_{E-\eta_1}^{E+\eta_1} (\operatorname{Re} \bar{m}(x+i3.5\kappa)) 3.5\kappa \chi(3.5\kappa) f_E'(x) dx \right| \\
&\quad + \frac{1}{\pi} \left| \int_{E-\eta_1}^{E+\eta_1} \int_{3.5\kappa}^{10\kappa} (\operatorname{Re} \bar{m}(x+iy)) \frac{\partial}{\partial y} (y \chi(y)) f_E'(x) dy dx \right| \\
&\leq \frac{1}{\pi} 2\eta_1 \left(\frac{1}{3.5\kappa} N^{-1+2\epsilon_0} \right) C_{ind} 3.5\kappa \frac{\|\psi'\|_\infty}{2\eta_1} + \frac{1}{\pi} \left| \int_{E-\eta_1}^{E+\eta_1} \int_{3.5\kappa}^{10\kappa} \left(\frac{1}{3.5\kappa} N^{-1+2\epsilon_0} \right) C_{ind} (\chi(y) + y \chi'(y)) \frac{\|\psi'\|_\infty}{2\eta_1} dy dx \right| \\
&\leq \frac{1}{\pi} C_{ind} \|\psi'\|_\infty N^{-1+2\epsilon_0} + \frac{2}{\pi} C_{ind} (1 + \|\chi'\|_\infty) \|\psi'\|_\infty N^{-1+2\epsilon_0} \\
&\leq M_5 N^{-1+2\epsilon_0}
\end{aligned}$$

for some $M_5 = M_5(V_p, \kappa, \epsilon_0) > 0$.

The estimations of $I-VIII$ complete the proof. \square

The following lemma is an analogue of Lemma 3.13 of [6].

Theorem 17.4. *Suppose $P_r > 20$ (thus $t_r > 3$). Suppose $\mathcal{L}(r)$ is true. Suppose $0.4P_r\epsilon_0 < \frac{3}{4}a_r$. For any $\alpha \in (0, 1/2)$, there exist*

$$M_8 = M_8(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0, \quad \text{and} \quad N_0 = N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$$

such that if $N > N_0$ and $k \in [\alpha N, (1-\alpha)N]$, then $|\gamma_k^{(N)} - \gamma_k| \leq M_8 N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0}$.

Proof. Set $\eta_1 = N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0}$. Suppose $\delta_* > 0$ depending on V_p and α such that $[\gamma_{0.5\alpha N}, \gamma_{(1-0.5\alpha)N}] \subset [c+3\delta_*, d-3\delta_*]$. So if $N > N_0(V_p, \epsilon_0, \alpha, a_r, t_r, P_r)$, then $[\gamma_{0.5\alpha N}-\eta_1, \gamma_{(1-0.5\alpha)N}+\eta_1] \subset [c+2\delta_*, d-2\delta_*]$.

According to Lemma 17.3, exist $M_6 = M_6(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ and $N_0 = N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$ and $E \in [\gamma_{0.5\alpha N}, \gamma_{(1-0.5\alpha)N}]$, then $E \pm \eta_1 \in [c+2\delta_*, d-2\delta_*]$ and

$$\left| \int f_{E \pm \eta_1}(t) \bar{\rho}(t) dt \right| \leq M_6 \cdot N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0}.$$

Thus when $N > N_0$ and $E \in [\gamma_{0.5\alpha N}, \gamma_{(1-0.5\alpha)N}]$,

$$\begin{aligned}
\int_{-\infty}^E \tilde{\rho}_1^{(N)}(t) dt &\leq \int_{\mathbb{R}} \tilde{\rho}_1^{(N)}(t) f_{E+\eta_1}(t) dt = \int_{\mathbb{R}} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) f_{E+\eta_1}(t) dt + \int_{\mathbb{R}} \tilde{\rho}(t) f_{E+\eta_1}(t) dt \\
&\leq M_6 \cdot N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} + \int_{\mathbb{R}} \tilde{\rho}(t) f_{E+\eta_1}(t) dt \\
&= M_6 \cdot N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} + \int_{-\infty}^E \tilde{\rho}(t) dt + \int_E^{E+2\eta_1} \tilde{\rho}(t) f_{E+\eta_1}(t) dt \\
&\leq \int_{-\infty}^E \tilde{\rho}(t) dt + M_6 \cdot N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} + 2\|\tilde{\rho}\|_{\infty} N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0},
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^E \tilde{\rho}_1^{(N)}(t) dt &\geq \int_{\mathbb{R}} \tilde{\rho}_1^{(N)}(t) f_{E-\eta_1}(t) dt = \int_{\mathbb{R}} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) f_{E-\eta_1}(t) dt + \int_{\mathbb{R}} \tilde{\rho}(t) f_{E-\eta_1}(t) dt \\
&\geq \int_{\mathbb{R}} \tilde{\rho}(t) f_{E-\eta_1}(t) dt - M_6 \cdot N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} \\
&= \int_{-\infty}^E \tilde{\rho}(t) dt - \int_{E-2\eta_1}^E \tilde{\rho}(t) (1 - f_{E-\eta_1}(t)) dt - M_6 \cdot N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} \\
&\geq \int_{-\infty}^E \tilde{\rho}(t) dt - M_6 \cdot N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} - 2\|\tilde{\rho}\|_{\infty} N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0},
\end{aligned}$$

and thus

$$\begin{aligned}
\left| \int_{-\infty}^E \tilde{\rho}_1^{(N)}(t) dt - \int_{-\infty}^E \tilde{\rho}(t) dt \right| &\leq M_6 \cdot N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} + 2\|\tilde{\rho}\|_{\infty} N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} \\
&\leq M_7 N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0}
\end{aligned} \tag{17.76}$$

for some $M_7 = M_7(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$

It is easy to see that there exists $s = s(V_p, \alpha) > 0$ such that if $k \in [\frac{\alpha}{2}N, (1 - \frac{\alpha}{2})N]$, then $\gamma_k \in [c+s, d-s]$. If $k \in [\alpha N, (1-\alpha)N]$, then according to (17.76), for every $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\begin{aligned}
\int_{-\infty}^{\gamma_{k+0.5\alpha N}} \tilde{\rho}_1^{(N)}(t) dt &= \int_{-\infty}^{\gamma_{k+0.5\alpha N}} \tilde{\rho}(t) dt + \int_{-\infty}^{\gamma_{k+0.5\alpha N}} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) dt \geq \frac{k}{N} + 0.5\alpha - M_7 N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} \geq \frac{k}{N}, \\
\int_{-\infty}^{\gamma_{k-0.5\alpha N}} \tilde{\rho}_1^{(N)}(t) dt &= \int_{-\infty}^{\gamma_{k-0.5\alpha N}} \tilde{\rho}(t) dt + \int_{-\infty}^{\gamma_{k-0.5\alpha N}} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) dt \leq \frac{k}{N} - 0.5\alpha + M_7 N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} \leq \frac{k}{N}
\end{aligned}$$

and thus $\gamma_k^{(N)} \in [\gamma_{k-0.5\alpha N}, \gamma_{k+0.5\alpha N}] \subset [\gamma_{0.5\alpha N}, \gamma_{(1-0.5\alpha)N}]$.

So for $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$ and $k \in [\alpha N, (1-\alpha)N]$,

$$\left| \int_{-\infty}^{\gamma_k^{(N)}} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) dt \right| = \left| \int_{\gamma_k^{(N)}}^{\gamma_k} \tilde{\rho}(t) dt \right| \geq |\gamma_k^{(N)} - \gamma_k| \min_{[c+s, d-s]} \tilde{\rho}(t)$$

and by (17.76) we have

$$|\gamma_k^{(N)} - \gamma_k| \leq \left(\min_{[c+s, d-s]} \tilde{\rho}(t) \right)^{-1} \left| \int_{-\infty}^{\gamma_k^{(N)}} (\tilde{\rho}_1^{(N)}(t) - \tilde{\rho}(t)) dt \right| \leq M_7 N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} \left(\min_{[c+s, d-s]} \tilde{\rho}(t) \right)^{-1}.$$

Setting $M_8 = M_7 \left(\min_{[c+s, d-s]} \tilde{\rho}(t) \right)^{-1}$ we complete the proof. \square

18 Rigidity at scale N^{-1} and the proof of Theorem 5.3

Recall that in Section 5 we defined $\{t_1, t_2, \dots\}$ and $\{P_1, P_2, \dots\}$ satisfying

1. $t_1 > P_1 > 0$.
2. For $k \in \{1, 2, \dots\}$, define $a_k = \frac{1}{2}(\frac{3}{4})^{k-1}$, $P_k = P_1 \times 0.2^{k-1}$ and $t_{k+1} = 2t_k + 1.6P_k$.

Suppose $r \in \{1, 2, \dots\}$. $\mathcal{L}(r)$ is defined (in Section 16) to be the following statement.

$\mathcal{L}(r)$: For any $\alpha \in (0, 1/2)$, there exist $N_0 = N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$ and $i \in [\alpha N, (1 - \alpha)N]$, then

$$\mathbb{P}^{\tilde{\mu}_s} \left(|\lambda_i - \gamma_i| > N^{-1+a_r+t_r\epsilon_0} \right) \leq \exp(-N^{P_r\epsilon_0}).$$

The following lemma is an analogue of Lemma 3.11 of [6].

Theorem 18.1. *Suppose $P_r > 20$ (thus $t_r > 3$) and $0.4P_r\epsilon_0 < \frac{3}{4}a_r$. Suppose $\mathcal{L}(r)$ is true. For any $\alpha \in (0, 1/2)$, there exist $N_0 = N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that for every $N > N_0$,*

$$\mathbb{P}^{\tilde{\mu}_s} \left(\exists k \in [\alpha N, (1 - \alpha)N] \text{ st. } |\lambda_k - \gamma_k| > N^{-1+\frac{3}{4}a_r+(2t_r+1.6P_r)\epsilon_0} \right) \leq \exp(-N^{0.2P_r\epsilon_0}).$$

Proof. According to Theorem 16.10 there exists $N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$ such that if $N > N_0$ and $k \in [\alpha N, (1 - \alpha)N]$ then we have

$$\mathbb{P}^{\tilde{\mu}_s} \left(\left| \lambda_k - \mathbb{E}^{\tilde{\mu}_s} \lambda_k \right| > N^{-1+\frac{a_r}{2}+(t_r+0.7P_r)\epsilon_0} \right) \leq \exp(-N^{0.2P_r\epsilon_0}). \quad (18.77)$$

Suppose $N > N_0$ and $k_1 \in [\alpha N, (1 - \alpha)N]$. Notice that $\lambda \mapsto \#\{i | \lambda_i \leq \gamma_{k_1}^{(N)} + \frac{1}{2}N^{-\frac{1}{2}+2\epsilon_0}\}$ is a symmetric function on $[a, b]^N$. So by Lemma 5.2,

$$\mathbb{E}^{\tilde{\mu}_s} \left[\#\{i | \lambda_i \leq \gamma_{k_1}^{(N)}\} \right] = \mathbb{E}^{\tilde{\mu}} \left[\#\{i | \lambda_i \leq \gamma_{k_1}^{(N)}\} \right] = \mathbb{E}^{\tilde{\mu}} \left[\sum_{i=1}^N \mathbf{1}_{(-\infty, \gamma_{k_1}^{(N)})}(\lambda_i) \right] \geq N \int_{-\infty}^{\gamma_{k_1}^{(N)}} \tilde{\rho}_1^{(N)}(t) dt = k_1. \quad (18.78)$$

On the other hand, according to (18.77), when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\begin{aligned} \mathbb{E}^{\tilde{\mu}_s} \left[\#\{i | \lambda_i < \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) - N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0}\} \right] &= \sum_{l=1}^N \mathbb{P}^{\tilde{\mu}_s}(\lambda_l < \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) - N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0}) \\ &\leq k_1 - 1 + (N - k_1 + 1) \mathbb{P}^{\tilde{\mu}_s}(\lambda_{k_1} < \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) - N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0}) \leq k_1 - 1 + (N - k_1 + 1) \exp(-N^{0.2P_r\epsilon_0}) \\ &\leq k_1 \end{aligned} \quad (18.79)$$

Similarly,

$$\begin{aligned}
\mathbb{E}^{\tilde{\mu}_s} \left[\# \{i | \lambda_i > \gamma_{k_1-1}^{(N)} - \frac{1}{2} N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0} \} \right] &= N - \mathbb{E}^{\tilde{\mu}} \left[\# \{i | \lambda_i \leq \gamma_{k_1-1}^{(N)} - \frac{1}{2} N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0} \} \right] \\
&= N - \mathbb{E}^{\tilde{\mu}} \left[\sum_{i=1}^N \mathbb{1}_{(-\infty, \gamma_{k_1-1}^{(N)} - \frac{1}{2} N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0})}(\lambda_i) \right] \\
&\geq N - N \int_{-\infty}^{\gamma_{k_1-1}^{(N)}} \tilde{\rho}_1^{(N)}(t) dt \\
&= N - (k_1 - 1).
\end{aligned} \tag{18.80}$$

and when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\begin{aligned}
\mathbb{E}^{\tilde{\mu}_s} \left[\# \{i | \lambda_i > \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) + N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0} \} \right] &= \sum_{l=1}^N \mathbb{P}^{\tilde{\mu}_s}(\lambda_l > \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) + N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0}) \\
&\leq N - k_1 + k_1 \mathbb{P}^{\tilde{\mu}_s}(\lambda_{k_1} > \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) + N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0}) \leq N - k_1 + k_1 \exp(-N^{0.2P_r\epsilon_0}) \\
&\leq N - k_1 + 1.
\end{aligned} \tag{18.81}$$

By (18.78) and (18.79),

$$\gamma_{k_1}^{(N)} \geq \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) - N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0}. \tag{18.82}$$

By (18.80) and (18.81),

$$\gamma_{k_1-1}^{(N)} - \frac{1}{2} N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0} \leq \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) + N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0}, \quad \text{i.e.,} \quad \gamma_{k_1-1}^{(N)} \leq \mathbb{E}^{\tilde{\mu}_s}(\lambda_{k_1}) + \frac{3}{2} N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0}. \tag{18.83}$$

According to Theorem 17.4 there exist

$$M_8 = M_8(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0,$$

$$N_0 = N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r) > 0$$

such that if $N > N_0$ and $k \in [\frac{1}{2}\alpha N, (1 - \frac{1}{2}\alpha)N]$, then $|\gamma_k^{(N)} - \gamma_k| \leq M_8 N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0}$.

Since $\tilde{\rho}(t) \neq 0$ for $t \in (c, d)$, there exists $s > 0$ depending on V_p, κ, ϵ_0 and α such that $\tilde{\rho}(t) > s$ if $t \in [\gamma_{\alpha N/2}, \gamma_{(1-(\alpha/2))N}]$. If $k \in [\alpha N, (1 - \alpha)N]$, then $k - 1 \in [\frac{1}{2}\alpha N, (1 - \frac{1}{2}\alpha)N]$ and

$$\frac{1}{N} = \int_{\gamma_{k-1}}^{\gamma_k} \tilde{\rho}(t) dt \geq |\gamma_k - \gamma_{k-1}| s,$$

so $|\gamma_k - \gamma_{k-1}| \leq \frac{1}{sN}$ and

$$|\gamma_k^{(N)} - \gamma_{k-1}^{(N)}| \leq |\gamma_k - \gamma_k^{(N)}| + |\gamma_k - \gamma_{k-1}| + |\gamma_{k-1} - \gamma_{k-1}^{(N)}| \leq 2M_8 N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} + \frac{1}{sN}. \tag{18.84}$$

So by (18.83) and (18.84), if $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$ and $k \in [\alpha N, (1 - \alpha)N]$, then

$$\begin{aligned}\mathbb{E}^{\tilde{\mu}_s}(\lambda_k) &\geq \gamma_{k-1}^{(N)} - \frac{3}{2}N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0} \geq \gamma_k^{(N)} - (2M_8N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} + \frac{1}{sN}) - \frac{3}{2}N^{-1+\frac{1}{2}a_r+(t_r+0.7P_r)\epsilon_0} \\ &\geq \gamma_k^{(N)} - 3M_8N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0}\end{aligned}$$

and according to (18.82)

$$|\mathbb{E}^{\tilde{\mu}_s}(\lambda_k) - \gamma_k^{(N)}| \leq 3M_8N^{-1+\frac{3}{4}a_r+(2t_r+1.5P_r)\epsilon_0} \quad (18.85)$$

According to Theorem 16.10, Theorem 17.4 and (18.85), when $N > N_0(V_p, \kappa, \epsilon_0, \alpha, a_r, t_r, P_r)$,

$$\begin{aligned}&\mathbb{P}^{\tilde{\mu}_s}(\exists k \in [\alpha N, (1 - \alpha)N] \text{ st. } |\lambda_k - \gamma_k| > N^{-1+\frac{3}{4}a_r+(2t_r+1.6P_r)\epsilon_0}) \\ &\leq \mathbb{P}^{\tilde{\mu}_s}(\exists k \in [\alpha N, (1 - \alpha)N] \text{ st. } |\lambda_k - \mathbb{E}^{\tilde{\mu}_s}(\lambda_k)| > \frac{1}{3}N^{-1+\frac{3}{4}a_r+(2t_r+1.6P_r)\epsilon_0}) \\ &+ \mathbb{P}^{\tilde{\mu}_s}(\exists k \in [\alpha N, (1 - \alpha)N] \text{ st. } |\mathbb{E}^{\tilde{\mu}_s}(\lambda_k) - \gamma_k^{(N)}| > \frac{1}{3}N^{-1+\frac{3}{4}a_r+(2t_r+1.6P_r)\epsilon_0}) \\ &+ \mathbb{P}^{\tilde{\mu}_s}(\exists k \in [\alpha N, (1 - \alpha)N] \text{ st. } |\gamma_k^{(N)} - \gamma_k| > \frac{1}{3}N^{-1+\frac{3}{4}a_r+(2t_r+1.6P_r)\epsilon_0}) \\ &\leq \exp(-N^{0.2P_r\epsilon_0}).\end{aligned}$$

□

Proof of Theorem 5.3. According to Theorem 15.1, Theorem 5.3 is true for $k = 1$. Suppose Theorem 5.3 is true for $k = k_0$. By Theorem 18.1, Theorem 5.3 is also true for $k = k_0 + 1$. □

A Initial estimate of μ : the proof of Lemma 2.3

In this section we prove Lemma 2.3.

Use L_N to denote the empirical measure: $L_N = \frac{1}{N} \sum_{i=1}^N \delta(x - \lambda_i)$. Use $M_1(\mathbb{R})$ to denote the set of probability measures on \mathbb{R} with topology induced by the Lipschitz metric:

$$d_{LU}(\mu_1, \mu_2) = \sup_{f \in \mathcal{F}_{LU}} \left| \int_{\mathbb{R}} f(x) d\mu_1(x) - \int_{\mathbb{R}} f(x) d\mu_2(x) \right|$$

where \mathcal{F}_{LU} is the set of Lipschitz functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with Lipschitz constant at most 1 and uniform bound 1.

Define $\Sigma : M_1(\mathbb{R}) \rightarrow \mathbb{R}$ and $I : M_1(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$\Sigma(\mu) = \begin{cases} \iint \ln|x - y| d\mu(x) d\mu(y) & \text{if } \int \ln(|x| + 1) d\mu(x) < \infty \\ -\infty & \text{otherwise} \end{cases}$$

$$I(\mu) = \begin{cases} \int V(x) d\mu(x) - \Sigma(\mu) - c^V & \text{if } \int V(x) d\mu(x) < \infty \\ +\infty & \text{otherwise} \end{cases}$$

where $c^V = \inf_{\nu \in M_1(\mathbb{R})} \left[\int V(x) d\nu(x) - \Sigma(\nu) \right]$.

Remark. Σ , I and c^V were introduced in Section 2.6.1 of [1].

According to Section 2.6.1, Appendix C.2 and Appendix D of [1] we have the following lemmas.

Lemma A.1. 1. $M_1(\mathbb{R})$ is a Polish space.

2. Σ , I and c^V are all well defined and $-\infty < c^V < +\infty$.

3. For any $\mu \in M_1(\mathbb{R})$ we have $\Sigma(\mu) \in [-\infty, +\infty)$ and $I(\mu) \in [0, +\infty]$.

4. For any $c > 0$, $\{\mu \in M_1(\mathbb{R}) | I(\mu) \leq c\}$ is a compact subset of $M_1(\mathbb{R})$.

For any $u > 0$, set $F_u = \{\mu \in M_1(\mathbb{R}) | d_{LU}(\mu, \rho(t)dt) \geq u\}$.

Lemma A.2. Suppose $u > 0$. Then $\mathbb{P}^\mu(L_N \in F_u) = \mathbb{P}^{\mu_s}(L_N \in F_u)$.

Proof. By definition and measure theory,

$$\begin{aligned}
& \mathbb{P}^\mu(L_N \in F_u) = 1 - \mathbb{P}^\mu(d_{LU}(L_N, \rho(t)dt) < u) \\
& = 1 - \mathbb{P}^\mu(L_N \in \bigcup_{i=1}^\infty \{ \sup_{f \in \mathcal{F}_{LU}} | \int_{\mathbb{R}} f(x) dL_N(x) - \int_{\mathbb{R}} f(x) \rho(x) dx | \leq u - \frac{1}{i} \}) \\
& = 1 - \lim_{i \rightarrow \infty} \mathbb{P}^\mu(L_N \in \{ \sup_{f \in \mathcal{F}_{LU}} | \int_{\mathbb{R}} f(x) dL_N(x) - \int_{\mathbb{R}} f(x) \rho(x) dx | \leq u - \frac{1}{i} \}) \\
& = 1 - \lim_{i \rightarrow \infty} \mathbb{P}^\mu(| \frac{1}{N} \sum_{k=1}^N f(\lambda_k) - \int_{\mathbb{R}} f(x) \rho(x) dx | \leq u - \frac{1}{i}, \forall f \in \mathcal{F}_{LU}). \tag{1.86}
\end{aligned}$$

Similarly,

$$\mathbb{P}^{\mu_s}(L_N \in F_u) = 1 - \lim_{i \rightarrow \infty} \mathbb{P}^{\mu_s}(| \frac{1}{N} \sum_{k=1}^N f(\lambda_k) - \int_{\mathbb{R}} f(x) \rho(x) dx | \leq u - \frac{1}{i}, \forall f \in \mathcal{F}_{LU}). \tag{1.87}$$

Since $(\lambda_1, \dots, \lambda_N) \mapsto | \frac{1}{N} \sum_{k=1}^N f(\lambda_k) - \int_{\mathbb{R}} f(x) \rho(x) dx |$ is a symmetric function, we have from Lemma 2.2 that

$$\begin{aligned}
& \mathbb{P}^\mu(| \frac{1}{N} \sum_{k=1}^N f(\lambda_k) - \int_{\mathbb{R}} f(x) \rho(x) dx | \leq u - \frac{1}{i}, \forall f \in \mathcal{F}_{LU}) \\
& = \mathbb{E}^\mu(\mathbb{1}_{(| \frac{1}{N} \sum_{k=1}^N f(\lambda_k) - \int_{\mathbb{R}} f(x) \rho(x) dx | \leq u - \frac{1}{i}, \forall f \in \mathcal{F}_{LU})}) \\
& = \mathbb{E}^{\mu_s}(\mathbb{1}_{(| \frac{1}{N} \sum_{k=1}^N f(\lambda_k) - \int_{\mathbb{R}} f(x) \rho(x) dx | \leq u - \frac{1}{i}, \forall f \in \mathcal{F}_{LU})}) \\
& = \mathbb{P}^{\mu_s}(| \frac{1}{N} \sum_{k=1}^N f(\lambda_k) - \int_{\mathbb{R}} f(x) \rho(x) dx | \leq u - \frac{1}{i}, \forall f \in \mathcal{F}_{LU}). \tag{1.88}
\end{aligned}$$

(1.86), (1.87), (1.88) together complete the proof. \square

Lemma A.3. For any $u > 0$ we have

$$\inf_{\mu \in F_u} I(\mu) > 0.$$

Proof. By definition

$$\begin{aligned} \inf_{\mu \in F_u} I(\mu) &= \inf_{\mu \in F_u} \left[\int V(x) d\mu(x) - \iint \ln |x - y| d\mu(x) d\mu(y) - c^V \right] \\ &= \frac{1}{2} \inf_{\mu \in F_u} \iint \left[V(x) + V(y) - 2 \ln |x - y| \right] d\mu(x) d\mu(y) \\ &\quad - \frac{1}{2} \inf_{\mu \in M_1(\mathbb{R})} \iint \left[V(x) + V(y) - 2 \ln |x - y| \right] d\mu(x) d\mu(y) \end{aligned}$$

By Lemma 2.1, $\rho(t)dt$ is the only minimizer (in $M_1(\mathbb{R})$) of

$$\mu \mapsto \iint \left[V(x) + V(y) - 2 \ln |x - y| \right] d\mu(x) d\mu(y).$$

Therefore we only need to show that there exists $\mu_0 \in F_u$ such that

$$\inf_{\mu \in F_u} \iint \left[V(x) + V(y) - 2 \ln |x - y| \right] d\mu(x) d\mu(y) = \iint \left[V(x) + V(y) - 2 \ln |x - y| \right] d\mu_0(x) d\mu_0(y)$$

or equivalently,

$$I(\mu_0) = \inf_{\mu \in F_u} I(\mu).$$

Set

$$A = \{\mu \in F_u | I(\mu) \leq 1 + \inf_{\mu \in F_u} I(\mu)\} \quad \text{and} \quad A_k = \{\mu \in F_u | I(\mu) \leq \frac{1}{k} + \inf_{\mu \in F_u} I(\mu)\} \quad (k = 1, 2, \dots).$$

Then A is a closed subset of $\{\mu \in M_1(\mathbb{R}) | I(\mu) \leq 1 + \inf_{\mu \in F_u} I(\mu)\}$ and A_k is a closed subset of $\{\mu \in M_1(\mathbb{R}) | I(\mu) \leq \frac{1}{k} + \inf_{\mu \in F_u} I(\mu)\}$. By Lemma A.1, A and every A_k are compact and closed.

If $\cap_{k=1}^{\infty} A_k = \emptyset$, then $\{A \setminus A_1, A \setminus A_2, \dots\}$ is a class of open subsets of A which covers A . Since A is compact, there exists $k_0 \in \mathbb{N}$ such that $\cup_{k=1}^{k_0} A \setminus A_k = A$. So $\cap_{k=1}^{k_0} A_k = \emptyset$ or equivalently, $A_{k_0} = \{\mu \in F_u | I(\mu) \leq \frac{1}{k_0} + \inf_{\mu \in F_u} I(\mu)\} = \emptyset$ which is a contradiction.

Therefore we proved that $\cap_{k=1}^{\infty} A_k \neq \emptyset$. It is easy to check that if $\mu_0 \in \cap_{k=1}^{\infty} A_k$, then

$$I(\mu_0) = \inf_{\mu \in F_u} I(\mu).$$

□

Lemma A.4. For any $\alpha_* > 0$, $\epsilon_* > 0$ and $1 \leq i \leq q$ there exist $N_0 > 0$ and $\tilde{\epsilon} > 0$ depending on V , α_* and ϵ_* such that if $N > N_0$ and

$$\exists k \in [(R_1 + \dots + R_{i-1} + \alpha_*)N, (R_1 + \dots + R_i - \alpha_*)N] \quad \text{with} \quad |\lambda_k - \eta_k| > \epsilon_*,$$

then $d_{LU}(L_N, \rho(t)dt) \geq \tilde{\epsilon}$.

Proof. Recall that $\rho(t) > 0$ on the interior of the support of ρ . So there must be $c > 0$ and $\delta > 0$ depending on V , α_* and ϵ_* such that

1. $\delta \leq \epsilon_*$
2. if $k \in [(R_1 + \dots + R_{i-1} + \alpha_*)N, (R_1 + \dots + R_i - \alpha_*)N]$ and $t \in [\eta_k - \delta, \eta_k + \delta]$, then $\rho(t) > c$.

Fix a natural number $N_0 > \frac{4}{c\delta}$. Suppose $N > N_0$ and $|\lambda_k - \eta_k| > \epsilon_*$ where $k \in [(R_1 + \dots + R_{i-1} + \alpha_*)N, (R_1 + \dots + R_i - \alpha_*)N]$.

If $\lambda_k < \eta_k - \epsilon_*$, set

$$f(x) = \begin{cases} 0 & \text{if } x \leq \eta_k - \delta \\ x - (\eta_k - \delta) & \text{if } x \in [\eta_k - \delta, \eta_k] \\ \delta & \text{if } x \geq \eta_k \end{cases}$$

It is easy to check that $f \in \mathcal{F}_{LU}$. We have

$$\int f(x)\rho(x)dx = \int_{\eta_k - \delta}^{\eta_k} (x - (\eta_k - \delta))\rho(x)dx + \delta \int_{\eta_k}^{+\infty} \rho(x)dx \geq \frac{1}{2}c\delta^2 + \delta(1 - \frac{k}{N})$$

and

$$\int f(x)L_N(x)dx = \frac{1}{N} \sum f(\lambda_i) \leq \frac{N+1-k}{N}\delta.$$

Thus $\int f(x)\rho(x)dx - \int f(x)L_N(x)dx \geq \frac{1}{2}c\delta^2 + \frac{\delta}{N} > \frac{1}{4}c\delta^2$ and $d_{LU}(L_N, \rho(t)dt) > \frac{1}{4}c\delta^2$.

If $\lambda_k > \eta_k + \epsilon_*$, set

$$g(x) = \begin{cases} 0 & \text{if } x \geq \eta_k + \delta \\ (\eta_k + \delta) - x & \text{if } x \in [\eta_k, \eta_k + \delta] \\ \delta & \text{if } x \leq \eta_k \end{cases}$$

Similarly we can prove that $g \in \mathcal{F}_{LU}$ and $\int g(x)\rho(x)dx - \int g(x)L_N(x)dx > \frac{1}{4}c\delta^2$. So we still have $d_{LU}(L_N, \rho(t)dt) > \frac{1}{4}c\delta^2$.

Setting $\tilde{\epsilon} = \frac{1}{4}c\delta^2$ we complete the proof. □

Proof of Lemma 2.3. According to Lemma A.4 and Lemma A.2, there exist $N_0 > 0$ and $\tilde{\epsilon} > 0$ depending on V , α_* and ϵ_* such that if $N > N_0$ then

$$\begin{aligned} \mathbb{P}^{\mu_s}(\exists k \in [(R_1 + \dots + R_{i-1} + \alpha_*)N, (R_1 + \dots + R_i - \alpha_*)N] \text{ such that } |\lambda_k - \eta_k| > \epsilon_*) \\ \leq \mathbb{P}^{\mu_s}(L_N \in F_{\tilde{\epsilon}}) = \mathbb{P}^{\mu}(L_N \in F_{\tilde{\epsilon}}) \end{aligned}$$

According to Theorem 2.6.1(c) of [1],

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \ln \mathbb{P}^{\mu}(L_N \in F_{\tilde{\epsilon}}) \leq - \inf_{\mu \in F_{\tilde{\epsilon}}} \left(\frac{\beta}{2} I(\mu) \right).$$

Since $\inf_{\mu \in F_{\tilde{\epsilon}}} I(\mu) > 0$, there must be $c > 0$ and $N_0 > 0$ such that when $N > N_0$,

$$\mathbb{P}^{\mu_s}(\exists k \in [(R_1 + \dots + R_{i-1} + \alpha_*)N, (R_1 + \dots + R_i - \alpha_*)N] \text{ such that } |\lambda_k - \eta_k| > \epsilon_*) \leq \exp(-cN^2).$$

□

B A Riemann-Hilbert problem and the proof of Lemma 8.1

In this section we use the method introduced in Chap 6 of [9] to prove Lemma 8.1.

The following theorem is well known.

Lemma B.1 (Sokhotski-Plemelj Theorem). *Let f be a complex-valued function which is defined and continuous on the real line, and let a and b be real constants with $a < 0 < b$. Then*

$$\lim_{\epsilon \rightarrow 0^+} \int_a^b \frac{f(x)}{x \pm i\epsilon} dx = \mp i\pi f(0) + P.V. \int_a^b \frac{f(x)}{x} dx.$$

Recall that:

1. V_p is analytic on $\Omega := \mathbb{C} \setminus ((-\infty, W_L] \cup [W_R, +\infty))$.
2. $W_L < a < b < W_R$.
3. $\tilde{\mu}$ is a probability measure on $[a, b]^N$:

$$\tilde{\mu}(dx) = \frac{1}{Z_{\tilde{\mu}}} e^{-\frac{N\beta}{2} \sum c_N V_p(x_i)} \prod_{i < j} |x_i - x_j|^\beta \prod_{i=1}^N \mathbb{1}_{[a,b]}(x_i) dx.$$

where $Z_{\tilde{\mu}}$ is the normalization constant and $|c_N - 1| < N^{-1+\epsilon_0}$.

4. the equilibrium measure of $\tilde{\mu}$ is $\tilde{\rho}(t) = \tilde{r}(t) \sqrt{(t-c)(d-t)} \mathbb{1}_{[c,d]}(t)$ where $r(t) > 0$ on $[c, d]$. Moreover, $\tilde{r}(t)$ has an analytic extension on a neighborhood of $[c, d]$.
5. $\tilde{m}(z)$ is the Stieltjes transform of $\tilde{\rho}(t)dt$: $\tilde{m}(z) = \int \frac{1}{z-t} \tilde{\rho}(t) dt$ ($z \notin [c, d]$).

To prove Lemma 8.1 it suffices to prove the following lemma

Lemma B.2. *$\tilde{r}(x)$ has an analytic extension $H(z)$ on Ω such that*

$$2\tilde{m}(z) - V_p'(z) = -2\pi H(z) \sqrt{(c-z)(d-z)} \quad \text{on } \Omega \setminus [c, d]$$

where $\sqrt{(c-z)(d-z)}$ is the branch such that $\sqrt{(c-z)(d-z)} \sim z$ as $z \rightarrow \infty$.

Lemma B.3. *Suppose \mathcal{L} is a clockwise contour in Ω such that $[c, d]$ is in the interior of \mathcal{L} and z is in the exterior of \mathcal{L} . We have*

$$\int_{\mathcal{L}} \frac{V_p'(\xi)}{\sqrt{(c-\xi)(d-\xi)}} \frac{d\xi}{\xi - z} = \frac{2}{i} \int_c^d \frac{V_p'(t)}{\sqrt{(t-c)(d-t)}} \frac{dt}{t - z}$$

where $\sqrt{(c-\xi)(d-\xi)}$ is the branch such that $\sqrt{(c-\xi)(d-\xi)} \sim \xi$ as $\xi \rightarrow \infty$.

Proof. This lemma is trivial. □

Lemma B.4. *Suppose $x \in \mathbb{R}$ and $W_L - x < u < v < W_R - x$ (thus if $r \in (u, v)$ then $r + x \in (W_L, W_R)$).*

Suppose $f(r) = \frac{V_p'(x+r)}{\sqrt{(r-u)(v-r)}}$ where $r \in (u, v)$. Then we have

$$\lim_{y \downarrow 0} \int_u^v \frac{r^2}{r^2 + y^2} \frac{f(r)}{r} dr = P.V. \int_u^v \frac{f(r)}{r} dr \quad \text{and} \quad \lim_{y \downarrow 0} \int_u^v \frac{yf(r)}{\pi(r^2 + y^2)} dr = f(0).$$

Proof. This lemma can be proved by using knowledge of harmonic analysis. \square

For $z \notin [c, d]$, define

$$G(z) = \frac{1}{\pi i} \int_c^d \frac{\tilde{\rho}(t)}{t-z} dt = \frac{i}{\pi} \tilde{m}(z) \quad \text{and} \quad \tilde{G}(z) = \frac{G(z)}{\sqrt{(c-z)(d-z)}}$$

where $\sqrt{(c-z)(d-z)}$ is the branch such that $\sqrt{(c-z)(d-z)} \sim z$ as $z \rightarrow \infty$.

According to (1-2) of [4], there exists $C_{max} > 0$ depending only on V_p such that

$$\begin{cases} 2 \int_c^d \rho(y) \ln |x-y| dy - V_p(x) \leq C_{max}, & \forall x \in [a, b] \\ 2 \int_c^d \rho(y) \ln |x-y| dy - V_p(x) = C_{max}, & \forall x \in [c, d]. \end{cases}$$

(In [4] the second line was only for a.s. $x \in [c, d]$, but here $x \mapsto \int_c^d \rho(y) \ln |x-y| dy$ is continuous because of the property of convolution.) So we have that

$$V'_p(x) = 2\text{P.V.} \int_c^d \frac{\tilde{\rho}(y)}{x-y} dy, \quad \forall x \in (c, d). \quad (2.89)$$

Lemma B.5. *If $z \in \Omega \setminus [c, d]$, then*

$$\tilde{G}(z) = \frac{1}{2\pi i} \int_c^d \frac{V'_p(x)}{\pi \sqrt{(x-c)(d-x)}} \frac{dx}{x-z}, \quad G(z) = \frac{\sqrt{(c-z)(d-z)}}{2\pi i} \int_c^d \frac{V'_p(x)}{\pi \sqrt{(x-c)(d-x)}} \frac{dx}{x-z}.$$

where $\sqrt{(c-z)(d-z)}$ is the branch such that $\sqrt{(c-z)(d-z)} \sim z$ as $z \rightarrow \infty$.

Proof. This lemma can be proved by using knowledge of harmonic analysis. \square

Now we prove Lemma B.2. Suppose $z \in \Omega \setminus [c, d]$. Suppose \mathcal{L} and \mathcal{C} are smooth clockwise contours on Ω such that

1. $z, [c, d]$ and \mathcal{L} are all in the interior of \mathcal{C} ;
2. $[c, d]$ is in the interior of \mathcal{L} and z is in the exterior of \mathcal{L} .

By Cauchy's integral theorem, we have

$$\frac{\sqrt{(c-z)(d-z)}}{4\pi i} \int_{\mathcal{C}} \frac{\frac{i}{\pi} V'_p(\xi)}{\sqrt{(c-\xi)(d-\xi)}} \frac{d\xi}{\xi-z} = -\frac{i}{2\pi} V'_p(z) + \frac{\sqrt{(c-z)(d-z)}}{4\pi i} \int_{\mathcal{L}} \frac{\frac{i}{\pi} V'_p(\xi)}{\sqrt{(c-\xi)(d-\xi)}} \frac{d\xi}{\xi-z}$$

where $\sqrt{(c-z)(d-z)}$ is the branch such that $\sqrt{(c-z)(d-z)} \sim z$ as $z \rightarrow \infty$ and so is $\sqrt{(c-\xi)(d-\xi)}$.

This together with Lemma B.3 and the definition of $G(z)$ give us

$$G(z) = \frac{i}{2\pi} V'_p(z) + \frac{\sqrt{(c-z)(d-z)}}{4\pi i} \int_{\mathcal{C}} \frac{\frac{i}{\pi} V'_p(\xi)}{\sqrt{(c-\xi)(d-\xi)}} \frac{d\xi}{\xi-z}$$

and

$$2\tilde{m}(z) - V'_p(z) = -\frac{\sqrt{(c-z)(d-z)}}{2} \int_{\mathcal{C}} \frac{\frac{i}{\pi} V'_p(\xi)}{\sqrt{(c-\xi)(d-\xi)}} \frac{d\xi}{\xi-z}$$

Suppose $x \in (c, d)$. According to Lemma B.1 and Lemma B.4,

$$\tilde{\rho}(x) = \operatorname{Re} \lim_{y \downarrow 0} G(x + iy) = \frac{\sqrt{(x-c)(d-x)}}{2\pi^2} \operatorname{P.V.} \int_c^d \frac{V'_p(t)}{\sqrt{(t-c)(d-t)}} \frac{dt}{t-x}$$

and

$$\tilde{r}(x) = \frac{1}{2\pi^2} \operatorname{P.V.} \int_c^d \frac{V'_p(t)}{\sqrt{(t-c)(d-t)}} \frac{dt}{t-x}.$$

Define

$$h(z) = \int_{\mathcal{O}} \frac{V'_p(\xi)}{\sqrt{(c-\xi)(d-\xi)}} \frac{d\xi}{\xi-z}$$

to be a function on Ω where \mathcal{O} is a clockwise smooth contour on Ω with z and $[c, d]$ in its interior. It is easy to see that h is analytic on Ω (actually we can directly compute its derivative).

By direct computation, for $x \in (c, d)$ and $z = x + iy$,

$$h(z) = -2\pi i \frac{V'_p(z)}{\sqrt{(c-z)(d-z)}} + \frac{2}{i} \int_c^d \frac{V'_p(t)}{\sqrt{(t-c)(d-t)}} \frac{dt}{t-z}$$

and

$$\lim_{y \downarrow 0} h(z) = -2i \operatorname{P.V.} \int_c^d \frac{V'_p(t)}{\sqrt{(t-c)(d-t)}} \frac{dt}{t-x}.$$

Therefore $H(z) := \frac{i}{4\pi^2} h(z)$ is an analytic extension of $\tilde{r}(x)$ on Ω . Furthermore,

$$2\tilde{m}(z) - V'_p(z) = -2\pi \sqrt{(c-z)(d-z)} H(z), \quad \forall z \in \Omega \setminus [c, d].$$

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